On spare parts optimization

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This manuscript deals with some mathematical optimization models for multi-level inventories of expensive repairable items. Early models in this area were developed and applied for the U.S. Air Force Logistics. Later, these early models have been extended in several directions, and used in a variety of civil application, see e.g. http://www.systecon.se/.

In the book "Optimal inventory modeling of systems: multi-echelon techniques" the author Craig Sherbrooke, who has developed several important model in this area, describes a part of the model cosidered in this manuscript as follows (in "aircraft language"): "When a malfunction is diagnosed on an aircraft, the malfuncioning item is removed from the aircraft and brought into base supply. If a spare is available, it is issued and installed on the aircraft; otherwise a backorder is established . . . which implies that there is a "hole" in an aircraft that causes it to be grounded . . . ".

We make frequent reference to the companion mini-compendium *On marginal allocation*, abbreviated MALLOC, which we assume that the reader has access to.

1. Model 1 (one base, one LRU)

We begin by considering the simplest model, which is characterized as follows:

1. There is only one *base*, with its own *inventory of spare items* and its own *workshop*.
2. There is only one organizational level, and thus no central depot.
3. Only one type of items is considered, here referred to as *aircraft engine*.
   This is a so called *line replaceable unit* , abbreviated LRU.

The rate at which aircrafts with a malfunctioning engine arrive at the base is modelled by a Poisson process with intensity λ engines per time unit.

The defect engine is immediately removed from the aircraft and brought into the workshop. If the inventory of functioning engines is non-empty, such an engine is immediately installed into the aircraft which is then operable again. But if the inventory of functioning engines is empty, a *backorder* is established and the aircraft is grounded and useless for the time beeing.

When a defect engine has been repaired in the workshop, it is immediately brought to the inventory of spare engines. The repair times are assumed to be independent and equally distributed random variables with expected value T time units. According to Palm's theorem, see Appendix, this implies that the number of engines in the workshop, at a randomly chosen time, is a Poisson random variable with expected value λT.

An important decision variable in the model is the following:

$s =$ the number of spare engines which has been purchased for the base, i.e. the number of engines in the inventory when there is no engine in the workshop.

At a given randomly chosen time, there are some natural random variables:

X = the number of engines currently in the workshop.
OH = the number of engines currently available in the inventory (on hand).
BO = the number of currently grounded aircrafts waiting for an engine (backorders).

Between these random variables, which can only take on non-negative integer values, the following relation holds:

$$BO - OH = X - s\,. \tag{1.1}$$

Moreover, at each time at least one of BO and OH is zero.
Therefore, BO and OH can be expressed as the following functions of X and s:

$$BO = (X - s)^+ = \max\{\,0\,,\; X - s\} \quad \text{and} \quad OH = (s - X)^+ = \max\{\,0\,,\; s - X\}. \tag{1.2}$$

1.1. Expected number of backorders in Model 1

As mentioned above, X is a Poisson random variable with expected value λT, *i.e.*,

$$p(k) = P(X = k) = \frac{(\lambda T)^k}{k\,!}\, e^{-\lambda T}. \tag{1.3}$$

A key quantity is the expected value of the number of back orders, i.e. the average number of aircrafts that are grounded while waiting for a working engine. This quantity can be expressed as $\mathrm{E}[BO] = \mathrm{E}[(X-s)^+]$, which from now on will be denoted $\mathrm{EBO}(s)$. Thus,

$$\mathrm{EBO}(s) = \mathrm{E}[BO] = \mathrm{E}[(X - s)^+]. \tag{1.4}$$

Since the probability distribution of X is given by (1.3), the computation of $\mathrm{EBO}(s)$ can be done recursively as follows.

First, $p(k)$ can be computed recursively, since $p(0) = e^{-\lambda T}$ and

$$p(k+1) = \frac{\lambda T}{k+1}\, p(k), \quad \text{for } k = 0, 1, 2, \ldots \tag{1.5}$$

Next, let $R(s) =$ the probability for shortage, i.e.

$$R(s) = P(X > s) = \sum_{k=s+1}^{\infty} p(k) \quad \text{for } s = 0, 1, 2, \ldots \tag{1.6}$$

$R(s)$ can also be computed recursively, since $R(0) = 1 - p(0)$ and

$$R(s+1) = R(s) - p(s+1), \quad \text{for } s = 0, 1, 2, \ldots \tag{1.7}$$

Further,

$$\mathrm{EBO}(0) = \mathrm{E}[(X-0)^+] = \mathrm{E}[X] = \lambda T\,, \tag{1.8}$$

while

$$\mathrm{EBO}(s) = \mathrm{E}[(X-s)^+] = \sum_{k=s+1}^{\infty} (k-s)p(k)\,, \tag{1.9}$$

and

$$\mathrm{EBO}(s+1) = \sum_{k=s+2}^{\infty} (k-s-1)p(k) = \sum_{k=s+1}^{\infty} (k-s-1)p(k)\,. \tag{1.10}$$

From (1.6), (1.9) and (1.10) the following simple recursion formula is obtained,

$$\mathrm{EBO}(s+1) = \mathrm{EBO}(s) - R(s), \quad \text{för } s = 0, 1, 2, \ldots \tag{1.11}$$

Assume that $\mathrm{EBO}(s)$ should be computed for $s = 0, 1, \ldots, s^{\max}$. This can easily be done using the following Matlab statements. (As the indexing of vectors in Matlab starts with 1, $p(0)$ above will be called p(1) in Matlab, etc.)

```
lamT = lambda*T;
p(1) = exp(-lamT);
R(1) = 1 - p(1);
EBO(1) = lamT;
for s=1:smax
  s1=s+1;
  p(s1) = lamT*p(s)/s;
  R(s1) = R(s) - p(s1);
  EBO(s1) = EBO(s) - R(s);
end
```

Note that since $p(s) > 0$ for all $s \geq 0$, it follows that $R(s+1) < R(s)$. Moreover,

$$\Delta\mathrm{EBO}(s) = \mathrm{EBO}(s+1) - \mathrm{EBO}(s) = -R(s) < 0\,, \text{ and} \tag{1.12}$$

$$\Delta\mathrm{EBO}(s+1) - \Delta\mathrm{EBO}(s) = p(s+1) > 0\,, \tag{1.13}$$

which means that $\mathrm{EBO}(s)$ is decreasing and *integer-convex*, see MALLOC.

1.2. An optimization problem in Model 1

We now consider the following possible optimization problem under Model 1:

$$\text{minimize} \quad f(s) = cs + q\,\text{EBO}(s)\,, \quad \text{subject to } s \in \{0, 1, 2, \ldots\}\,. \tag{1.14}$$

where the constant $c > 0$ can be interpreted (approximately) as the cost for a spare engine, while the constant $q > 0$ can be interpreted (approximately) as the cost for an aircraft:
If the average number of grounded aircrafts is decreased by m, then m aircrafts may be sold while maintaining the same operational capacity.

Let $\Delta f(s) = f(s+1) - f(s)$. Then

$$\Delta f(s) = c + q\,\Delta\text{EBO}(s) = c - qR(s)\,. \tag{1.15}$$

Since $q > 0$ and $\text{EBO}(s)$ is integer-convex, it follows that $f(s)$ is integer-convex, and then the following proposition follows from Prop 1.1 in MALLOC.

Prop 1.1: Let $f(s) = cs + q\,\text{EBO}(s)$, for $s \in \{0, 1, 2, \ldots\}$. Then

$$\hat{s} = 0 \text{ minimizes } f(s) \text{ if and only if } R(0) \le \frac{c}{q}\,, \tag{1.16}$$

$$\hat{s} > 0 \text{ minimizes } f(s) \text{ if and only if } R(\hat{s}) \le \frac{c}{q} \le R(\hat{s} - 1)\,. \tag{1.17}$$

A simple algorithm for solving problem (1.14) is then to calculate $R(s)$ for $s = 0, 1, 2, \ldots$ until an $\hat{s}$ is found such that (for the first time) $R(\hat{s}) \le c/q$. Then $\hat{s}$ is an optimal solution.

2. Model 2 (one base, several LRU)

In this model, we extend Model 1 to the case that there are several different line replaceable units (LRU) in each aircraft. More precisely, we assume that there are $n > 1$ different LRU, here referred to as $\mathrm{LRU}_1, \dots, \mathrm{LRU}_n$. As soon as any of these is malfunctioning, it must be replaced by a functioning one before the aircraft can be used again. The assumptions and notations from Model 1 (which corresponds to $n = 1$) are then generalized as follows.

Aircrafts with defect LRU_j arrive to the base according to a Poisson process with intensity λ_j. The repair times for LRU_j are assumed to be independent and equally distributed random variables with expected value T_j.

The important decision variables in the model are the integers $s_1, \dots, s_n$, where

s_j = the number of spare units of LRU_j which have been purchased for the base, i.e. the number of LRU_j in the inventory when there is no LRU_j in the workshop.

Further, let $\mathbf{s} = (s_1, \dots, s_n)^\mathsf{T}$.
Let c_j be the cost per spare unit of LRU_j, and let $\mathbf{c} = (c_1, \dots, c_n)^\mathsf{T}$.
Let $C(\mathbf{s}) = \mathbf{c}^\mathsf{T}\mathbf{s}$ = the total cost of spare units at the base.

Consider the system at a given randomly chosen time, and let
X_j = the number of LRU_j in the workshop.

According to Palm's theorem, X_j has a Poisson distribution with expected value $\lambda_j T_j$, i.e.

$$p_j(k) = P(X_j = k) = \frac{(\lambda_j T_j)^k}{k\,!}\, e^{-\lambda_j T_j}. \tag{2.1}$$

Let EBO($\mathbf{s}$) be the average number of aircrafts grounded due to shortage of some LRU. Then

$$\mathrm{EBO}(\mathbf{s}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_j) = \sum_{j=1}^{n} \mathrm{E}[(X_j - s_j)^+], \tag{2.2}$$

where $\mathrm{EBO}_j(s_j)$ is the average number of aircrafts grounded due to shortage of LRU_j.
As in Model 1, it holds that

$$\mathrm{EBO}_j(s_j + 1) = \mathrm{EBO}_j(s_j) - R_j(s_j)\,, \tag{2.3}$$

where

$$R_j(s_j) = P(X_j > s_j) = \sum_{k=s_j+1}^{\infty} p_j(k)\,. \tag{2.4}$$

Consequently, we obtain recursive equations of the same type as in Model 1.

Now let $S = \{\mathbf{s} = (s_1, \dots, s_n)^\mathsf{T} \mid s_j \in \{0, 1, 2, \dots\} \text{ for all } j\}$.

S is an infinite set. In practice, the set S can be made finite by only considering the points in S that satisfy $C(\mathbf{s}) \le C^{\max}$, where $C^{\max}$ is a upper limit for how much the spare parts can possibly be allowed to cost. However, the number of elements in S is typically extremely large, for realistic values of n and $C^{\max}$.

2.1. Efficient solutions of Model 2

Each $\mathbf{s} \in S$ induces a spare parts cost $C(\mathbf{s})$ and an average number of backorders $\mathrm{EBO}(\mathbf{s})$. The vector $\hat{\mathbf{s}} \in S$ is an *efficient solution* and $(C(\hat{\mathbf{s}}), \mathrm{EBO}(\hat{\mathbf{s}}))$ is an *efficient point* if there is a constant $q > 0$ such that $\hat{\mathbf{s}}$ is an optimal solution to the following optimization problem in $\mathbf{s}$:

$$\text{minimize} \;\; C(\mathbf{s}) + q\,\mathrm{EBO}(\mathbf{s}) \;\; \text{subject to } \mathbf{s} \in S\,. \tag{2.5}$$

The following geometrical interpretation of the efficient points was provided in MALLOC: Let $M = \{(C(\mathbf{s}), \mathrm{EBO}(\mathbf{s})) \mid \mathbf{s} \in S\}$ and assume that all the points in M are plotted in a coordinate system where the horizontal axis shows $C(\mathbf{s})$ and the vertical axis shows $\mathrm{EBO}(\mathbf{s})$. The convex hull of M is defined as the smallest convex set that contains the whole set M. The efficient curve corresponding to the set M is the piecewise linear curve which constitutes the "southwestern boundary" of the convex hull of M.
Points $(C(\mathbf{s}), \mathrm{EBO}(\mathbf{s})) \in M$ which lie on the efficient curve are the efficient points, and the corresponding vectors $\mathbf{s} \in S$ are the efficient solutions.

The following three propositions are immediate consequences of Prop 3.1–3.3 in MALLOC:

Prop 2.1: $\hat{\mathbf{s}} \in S$ minimizes $C(\mathbf{s}) + q\,\mathrm{EBO}(\mathbf{s})$ if and only if the following conditions are satisfied for each $j = 1, \ldots, n$:

$$\frac{R_j(0)}{c_j} \le \frac{1}{q} \quad \text{if} \;\; \hat{s}_j = 0\,, \tag{2.6}$$

$$\frac{R_j(\hat{s}_j)}{c_j} \le \frac{1}{q} \le \frac{R_j(\hat{s}_j - 1)}{c_j} \quad \text{if} \;\; \hat{s}_j > 0\,. \tag{2.7}$$

Prop 2.2: $\hat{\mathbf{s}} \in S$ is an efficient solution and $(C(\hat{\mathbf{s}}), \mathrm{EBO}(\hat{\mathbf{s}})) \in M$ is an efficient point if and only if there is a constant $q > 0$ such that the conditions (2.6)–(2.7) are satisfied for each $j = 1, \ldots, n$.

Prop 2.3: Assume that $\hat{\mathbf{s}} \in S$ is an efficient solution and let $\widehat{C} = C(\hat{\mathbf{s}})$ and $\widehat{\mathrm{EBO}} = \mathrm{EBO}(\hat{\mathbf{s}})$. Then $\hat{\mathbf{s}}$ is an optimal solution to both the following optimization problems:

$$\text{minimize} \;\; C(\mathbf{s}) \;\; \text{subject to} \;\; \mathrm{EBO}(\mathbf{s}) \le \widehat{\mathrm{EBO}}\,, \;\; \mathbf{s} \in S\,. \tag{2.8}$$

$$\text{minimize} \;\; \mathrm{EBO}(\mathbf{s}) \;\; \text{subject to} \;\; C(\mathbf{s}) \le \widehat{C}\,, \;\; \mathbf{s} \in S\,. \tag{2.9}$$

2.2. Marginal Allocation Algorithm for Model 2

We now describe a surprisingly simple algorithm for determining the efficient curve. The algorithm generates efficient solutions $\mathbf{s}^{(1)}, \mathbf{s}^{(2)}, \mathbf{s}^{(3)}, \ldots$ "from left to right", *i.e.*, each new generated point has a higher value on $C(\mathbf{s})$ but a lower value on $\mathrm{EBO}(\mathbf{s})$ than the previously generated point. Throughout the algorithm $\mathbf{s}^{(k)}$ denotes the k:th generated efficient solution, $C^{(k)}$ denotes the corresponding spare part cost $C(\mathbf{s}^{(k)})$, and $\mathrm{EBO}^{(k)}$ denotes the corresponding expected number of backorders $\mathrm{EBO}(\mathbf{s}^{(k)})$. The algorithm terminates when there is no longer any efficient solution with $C(\mathbf{s}) \leq C^{\max}$.

Step 0:
Generate a table with n columns as follows. For $j = 1, \ldots, n$, fill the j:th column from the top and down with the quotients $R_j(0)/c_j$, $R_j(1)/c_j$, $R_j(2)/c_j$, etc. (A moderate number of quotients will suffice, additional quotients can be calculated as needed.) Note that the quotients are positive and strictly decreasing in each column.
Set $k = 0$, $\mathbf{s}^{(0)} = (0, \ldots, 0)^\mathsf{T}$, $C^{(0)} = 0$ and $\mathrm{EBO}^{(0)} = \sum_{j=1}^{n} \lambda_j T_j$.
Let all the quotients in the table be *uncanceled*.

Step 1:
Select the largest uncanceled quotient in the table (if there are several equally large, choose one of these arbitrarily). Cancel this quotient and let ℓ be the number of the column from which the quotient was canceled.

Step 2:
Let $k := k + 1$. Then let $s_\ell^{(k)} = s_\ell^{(k-1)} + 1$ and $s_j^{(k)} = s_j^{(k-1)}$ for all $j \neq \ell$.

Further, calculate $C^{(k)} = C^{(k-1)} + c_\ell$ and $\mathrm{EBO}^{(k)} = \mathrm{EBO}^{(k-1)} - R_\ell(s_\ell^{(k-1)})$.

If $C^{(k)} \geq C^{\max}$, terminate the algorithm. Otherwise, go to Step 1.

2.3. Some properties of the algorithm

Note that each generated solution $\mathbf{s}^{(k)}$ differs from the previously generated solution $\mathbf{s}^{(k-1)}$ in just one component. The name of the algorithm stems from the fact that

$$\frac{R_j(s_j)}{c_j} = \frac{-\Delta \mathrm{EBO}_j(s_j)}{c_j} = \frac{\text{decrease in EBO}(\mathbf{s}) \text{ if } s_j \text{ is increased by 1}}{\text{increase in } C(\mathbf{s}) \text{ if } s_j \text{ is increased by 1}}$$

Hence, in each step of the algorithm, we increase the s_j which gives marginally the largest reduction of $\mathrm{EBO}(\mathbf{s})$ per invested crown.

The following two propositions are immediate consequences of Prop 3.1 and 3.2 in MALLOC:

Prop 2.3: Each generated solution $\mathbf{s}^{(k)}$ is an efficient solution.

Prop 2.4: Assume that all quotients $R_j(s_j)/c_j$ in the original table are different.
Then the algorithm generates all efficient solutions which satisfy $C(\mathbf{s}) \leq C^{\max}$.

3. Model 3, the METRIC model (a central depot and several bases)

In this METRIC model (Multi-Echelon Technique for Recoverable Item Control), there are two organizational levels, but (to begin with) only one type of LRU, which is again referred to as *aircraft engine*. On the lowest organizational level there are n *bases*, each equipped with a *local inventory* of spare engines but *no workshop*. On the highest level there is a central *depot* with a *central workshop* and a *central inventory* of spare engines.

Decision variables in the model are:
s_j = the number of spare engines at base j, for $j = 1, \ldots, n$, and
s_0 = the number of spare engines at the depot.

At base j, aircrafts with malfunctioning engines arrive according to a Poisson process with intensity λ_j. When a defect engine arrives at the base it is immediately replaced by a functioning engine from the local inventory of spare engines, unless this is empty.

If there is no engine in the local inventory that can replace the defect engine a *backorder* is established at the base, and the corresponding aircraft is grounded.

The defect engine is sent directly to the central workshop. At the same time, a functioning engine is sent from the central inventory to the local inventory at the base. But if the central inventory is empty, so that no engine can be sent to the base, a *depot backorder* is established. This does not necessarily implies that an aircraft is grounded, but the risk of backorders at the bases increases.

The transportation time T_{bd} for a defect engine from a base to the central depot is assumed to be deterministic and known, and the same is assumed for the transportation time T_{db} for a functioning engine from the central depot to a base. For simplicity, we assume that there is no difference between the bases in this respect.

The repair time for a defect engine at the central workshop is assumed to be a random variable with expected value T_{rep}. An important assumption (approximation) in the model is that these repair times are independent and equally distributed.

The question now is how large the inventories of spare engines should be chosen, both locally at the bases and centrally at the depot. In particular, we are interested in determining the efficient curve which relates the cost of spare engines (horizontal axis) to the average number of grounded aircrafts (vertical axis) when the spare engines are allocated in an optimal way.

3.1. Analysis of the situation at the depot

Let X_0 = the number of defect engines that are in, or on their way to, the workshop.

From the given conditions, it follows that defect engines arrive to the workshop according to a Poisson process with intensity $\lambda_0 = \lambda_1 + \ldots + \lambda_n$. As the repair times have been assumed independent, it follows from Palm's theorem that X_0 is a Poisson random variable with

$$\mathrm{E}[X_0] = \lambda_0 T_0, \quad \text{where} \quad T_0 = T_{bd} + T_{rep}. \tag{3.1}$$

This implies that it is possible to compute $\mathrm{EBO}_0(s_0) = \mathrm{E}[(X_0 - s_0)^+]$, *i.e.*, the average number of depot backorders, with the same type of recursive equations as were used in Model 1.

3.2. Analysis of the situation at a base

Let X_j = the number of engines in the *pipeline* at base j, *i.e.*, the number of defect engines that have been sent from base j to the central workshop, but for which replacement engines have still not been delivered to the local inventory at base j.

Then $X_j = Y_j + Z_j$, where

Y_j = the number of defect engines which have arrived at base j during the last T_{db} time units,

Z_j = the number of defect engines which arrived at base j more than T_{db} time units ago, but which were depot backorders T_{db} time units ago.

Since Y_j is the number of Poisson arrivals in a given time interval, Y_j is a Poisson random variable with $\mathrm{E}[Y_j] = \lambda_j T_{db}$.

Let $Z_0 = Z_1 + \ldots + Z_n$ = the total number of depot backorders T_{db} time units ago.
Note that Z_0 has the same distribution as $(X_0 - s_0)^+$, so that

$$\mathrm{E}[Z_0] = \mathrm{E}[(X_0 - s_0)^+] = \mathrm{EBO}_0(s_0). \tag{3.2}$$

Since Z_j is the part of Z_0 that corresponds to base j, and since, on average, λ_j/λ_0 of all the defect engines at the central workspace originate from base j, we obtain

$$\mathrm{E}[Z_j] = \frac{\lambda_j}{\lambda_0}\mathrm{E}[Z_0] = \frac{\lambda_j}{\lambda_0}\mathrm{EBO}_0(s_0), \tag{3.3}$$

Thus, we get the following expressions for the expected value of the number of engines in the pipeline for base j:

$$\mathrm{E}[X_j] = \mathrm{E}[Y_j + Z_j] = \lambda_j(T_{db} + \frac{\mathrm{EBO}_0(s_0)}{\lambda_0}). \tag{3.4}$$

We now make the assumption (approximation) that the pipeline times of the engines at a base are *independent* and equally distributed random variables. The pipeline time for an engine is defined as the time *from* the defect engine arrived at a base and is sent to the central workshop *until* the local inventory at the base has received a corresponding functioning engine in exchange.

From Palm's theorem it follows that X_j is a Poisson random variable with expected value given by (3.4) above, i.e.,

$$p_j(k) = P(X_j = k) = \frac{(\lambda_j T_j)^k}{k\,!}\,e^{-\lambda_j T_j}, \quad \text{where} \quad T_j = T_{db} + \frac{\mathrm{EBO}_0(s_0)}{\lambda_0}. \tag{3.5}$$

This means that it is possible to compute $\mathrm{E}[(X_j - s_j)^+]$, *i.e.*, the average number of back orders at base j, with the same type of recursive equations that was used in Model 1.
Note that $\mathrm{E}[(X_j - s_j)^+]$ also depends on s_0, since T_j depends on s_0. We will therefore use the notation

$$\mathrm{EBO}_j(s_0, s_j) = \mathrm{E}[(X_j - s_j)^+]. \tag{3.6}$$

4. Marginal Allocation Algorithm for Model 3

Let $\mathbf{s} = (s_1, \ldots, s_n)^{\mathsf{T}}$, and let $\mathrm{EBO}(s_0, \mathbf{s})$ = the average number of grounded aircrafts, i.e.,

$$\mathrm{EBO}(s_0, \mathbf{s}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_0, s_j)\,. \tag{4.1}$$

Let c = the cost per spare engine, and let $C(s_0, \mathbf{s})$ = the total cost of the spare engines, i.e.,

$$C(s_0, \mathbf{s}) = c\,s_0 + \sum_{j=1}^{n} c\,s_j. \tag{4.2}$$

It will now be described how to determine the efficient curve, in a coordinate system where the horizontal axis shows $C(s_0, \mathbf{s})$ and the vertical axis shows $\mathrm{EBO}(s_0, \mathbf{s})$. Only efficient solutions with $C(s_0, \mathbf{s}) \le C^{\max}$ will be considered, where $C^{\max}$ is a upper bound on the possible cost for spare engines. Equivalently, this can be expressed as $s_0 + \sum_{j=1}^{n} s_j \le s^{\max}$, where $s^{\max}$ is the largest integer such that $c\,s^{\max} \le C^{\max}$.

4.1. Algorithm for a fixed s_0

In this section it is assumed that s_0 is held fixed (to a non-negative integer).

Then it is possible, by using the marginal allocation algorithm of Model 2, to determine the efficient solutions for allocation of spare engines to the bases. More precisely, first $\mathrm{EBO}_0(s_0)$ is calculated and then the algorithm in section 2.2 is applied with the following modifications:
– the index j now corresponds to base number j (and not LRU_j),
– the cost coefficients c_j are now all equal to c (the cost for a spare engine),
– the time constants T_j are now all equal to $T_{db} + \mathrm{EBO}_0(s_0)/\lambda_0$ (the expected pipeline times).

The results from the algorithm will be a set of efficient points $\mathbf{s}^{(k)}$, the corresponding expected number of grounded aircrafts,

$$\mathrm{EBO}(s_0, \mathbf{s}^{(k)}) = \sum_{j=1}^{n} \mathrm{EBO}_j(s_0, s_j^{(k)})\,, \tag{4.3}$$

and the corresponding total cost of spare engines,

$$C(s_0, \mathbf{s}^{(k)}) = c\,s_0 + \sum_{j=1}^{n} c\,s_j^{(k)} = c\,s_0 + c\,k, \tag{4.4}$$

where the last equality follows from the fact that in each iteration of the marginal allocation algorithm exactly one more spare engine is allocated, and $\mathbf{s}^{(0)} = (0, \ldots, 0)^{\mathsf{T}}$. This means that it is sufficient to calculate $\mathbf{s}^{(k)}$ for $k = 0, 1, \ldots, s^{\max} - s_0$.

The generated efficient solutions are saved, and also the following EBO-values:

$$F(s_0,k) = \text{EBO}(s_0,\mathbf{s}^{(k)}), \text{ for } k = 0,1,\ldots,s^{\max} - s_0. \tag{4.5}$$

4.2. The complete algorithm for Model 3

Start with $s_0 = 0$, and apply the algorithm described above (for fixed s_0).
This gives a set of efficient solutions for the case $s_0 = 0$, and corresponding EBO-values:

$$F(0,0),\ F(0,1),\ \ldots,\ F(0,s^{\max}). \tag{4.6}$$

The restricted efficient curve for the case $s_0 = 0$ is then the piecewise linear curve between the $s^{\max}+1$ points

$$(0\,,\ F(0,0)),\ (c\,,\ F(0,1)),\ (2c\,,\ F(0,2)),\ \ldots,\ (s^{\max}c\,,\ F(0,s^{\max}). \tag{4.7}$$

Then let $s_0 = 1$, and apply the algorithm described above (for fixed s_0).
This gives a set of efficient solutions for the case $s_0 = 1$, and corresponding EBO-values:

$$F(1,0),\ F(1,1),\ldots,\ F(1,s^{\max}-1). \tag{4.8}$$

The restricted efficient curve for the case $s_0 = 1$ is then the piecewise linear curve between the $s^{\max}$ points

$$(c\,,\ F(1,0)),\ (2c\,,\ F(1,1)),\ \ldots,\ (s^{\max}c\,,\ F(1,s^{\max}-1)). \tag{4.9}$$

This is repeated for $s_0 = 2,\ldots,s^{\max}$.
Note that the restricted efficient curve for the case $s_0 = s^{\max}$ consists of a single point $(s^{\max}c\,,\ F(s^{\max},0))$.

We have now obtained $s^{\max}+1$ curves, each corresponding to a fixed value on s_0.
A natural curve for the complete model, where s_0 is not fixed, is then the piecewise linear curve between the $s^{\max}+1$ points

$$(0\,,\ F(0)),\ (c\,,\ F(1)),\ (2c\,,\ F(2)),\ \ldots,\ (s^{\max}c\,,\ F(s^{\max}), \tag{4.10}$$

where

$$F(\ell) = \min_{s_0}\{\ F(s_0,\ \ell - s_0) \mid 0 \le s_0 \le \ell\ \},\quad \ell = 0,1,\ldots,s^{\max}. \tag{4.11}$$

Note that $F(\ell)$ is the minimum value of $\text{EBO}(s_0,\mathbf{s})$ if $C(s_0,\mathbf{s})$ is required to be $\le c\,\ell$.
If this curve is convex, then it is also the efficient curve for Model 3. Otherwise, the efficient curve is obtained by generating the southwestern boundary of the convex hull of the $s^{\max}+1$ points in (4.10). This is an easy task.

4.3. Several LRU in Model 3

It is easy to extend the model above to cover the case where there are several different types of LRU:s, denoted $\text{LRU}_1, \ldots, \text{LRU}_m$. In that case, we consider one LRU_j at a time, and determine the efficient curve for each LRU_j by using the method described above. This results in m curves. Thereafter, we determine a total efficient curve including all the LRU_j:s. This is again done by marginal allocation: The total efficient curve is constructed from left to right by line segments from the m curves of the individual LRU_j; first the steepest segment is used, then the second-steepest, etc.

5. APPENDIX: Palm's Theorem

Theorem: Assume that defect items arrive to a workshop according to a Poisson process with an intensity of λ items per time unit. Furthermore, assume that the repair times for the defect items are independent, equally distributed random variables with expected value T time units. Then the number of defect items in the workshop is a Poisson random variable with expected value λT items.

Remark: The "repair time" is the time *from* a defect item arrives at the workshop *until* the same item has been repaired and leaves the workshop.

Sketch of Proof:

Let τ denote the repair time for a defect item. We will only prove the theorem for the case where τ is a discrete random variable with finite sample space $\{t_1, \ldots, t_N\}$ (which can be arbitrarily dense). Assume that $t_1, \ldots, t_N$ are known and that $p_i = P(\tau = t_i)$ are the corresponding probabilities which are also known and satisfy $\sum p_i = 1$ and $\sum p_i t_i = T$.

We can then consider the situation as follows: Defect items arrive according to a Poisson process with intensity λ. For each arriving item, the length of the repair time is decided by a random trial. If the result of the trial is that the repair time should be t_i, which occurs with probability p_i, then the item is placed in the i:th *sub-workshop* which has a deterministic repair time $= t_i$. When the item leaves this sub-workshop t_i time units later, it leaves the real workshop as well.

It is a well known property of Poisson processes that the above procedure leads to that defect items arrive to the N different sub-workshops according to independent Poisson processes with intensities $\lambda_i = p_i \lambda$, for $i = 1, \ldots, N$.

Let $X_i =$ the number of items in the i:th sub-workshop. Then $X_i =$ the number of items that arrived to the i:th sub-workshop during the last t_i time units. According to another well-known property of Poisson processes, this number is a Poisson distributed random variable with expected value $\lambda_i t_i$.

Let $X =$ the number of items in the real workshop. Since each arriving item is placed in one sub-workshop, it follows that $X = \sum X_i$, where the X_i are independent Poisson random variables. According to a well-known property of Poisson distributions, this implies that X is a Poisson random variable with expected value $\sum \lambda_i t_i = \lambda \sum p_i t_i = \lambda T$, which is what we wanted to show.