KTH Mathematics

Exam in SF2863 Systems Engineering
Saturday December 18, 2010, 08.00–13.00

Examiner: Per Enqvist, tel. 790 62 98

Allowed tools: Pen/pencil, ruler and eraser. A formula sheet is handed out.

No calculator is allowed!

Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Explain carefully the methods you use, in particular if you use a method not taught in the course. Conclusions should always be motivated.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each sheet!
25 points, including bonus points from the homeworks, is sufficient for passing the exam. For 23-24 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam have been reported. Contact the examiner as soon as possible for such a completion.
The order of the exercises does not correspond to their difficulty level.

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1. Frasse has started a new life as an agent at ESA, and he approaches every thing in his life with an ambition of perfection.

His first problem when going to work is to find a good parking space. The office is located along a long road, with T parking spaces in sequence before the office, one space right in front of the office, and T after the office. There are $2T+1$ spots, and we can number them as $t=-T$ for the first one and $t=T$ for the last one. The value for parking in spot t is given by $|t|$. Each spot can either be full or empty, and we assume that the probability of a spot being empty is p, and that this probability does not depend on if the other spots are full or empty. (Hence the probability for a spot being full is $q=1-p$)

If Frasse finds an empty spot he can decide to park there, or take the chance to continue and not be able to turn back. He is not able to see in advance if there are any empty parking spots there. If he ends up without a spot, he returns home in utter disappointment, and if this should happen the value function is $M>T$.

(a) Help Frasse to minimize the expected value for his parking. Determine a recursive equation for the optimal value function with boundary constraints. Define carefully your states and describe how you can determine the optimal strategy for when to take an empty parking spot or not.
. . . . (4p)

(b) To practice before going to the big office, and to verify his algorithm by simulation, Frasse builds a small lego replica of the parking street with $p=0.5$, $T=2$ and $M=5$. Use the recursion from (a) to determine the smallest possible expected value and the parking strategy he should use to obtain it. . . . . . (4p)

2. Frasses first assignment at ESA is to handle the inventory of coffee for the whole office.

(a) As a first model Frasse approximates the demand for coffee as constant 24 hours per day and 7 days per week at a level of 10 kilos per hour. Shortage of coffee is not allowed. Each time he orders coffee from their local supplier they have to pay 100 Euro as ordering cost, 5 Euro per kilo coffee and the coffee is assumed to be delivered as ordered. The holding cost for the coffee is 0.00002 Euro per hour and kilo.

What is the optimal order quantity?
(minimizing total cost per time unit)
How often should Frasse order this amount of coffee? .... (4p)

(b) After the first 5 weeks, and a number of harsh complaints, Frasse realizes he has to revise his model. Most of the complaints were that the coffee was too old for the high quality minded workers at ESA and their sophisticated taste. Therefore, Frasse has been instructed to buy new coffee every week.

Based on data from the first 5 weeks he decides to model the demand of coffee for one week as a stochastic variable D with probability density function

$$f_D(d) = \begin{cases} 1/400 & \text{if } d \in [1500\ 1900] \\ 0 & \text{otherwise} \end{cases}$$

If there is coffee left at the end of the week he sells the excess coffee to a less demanding customer for 2 Euro per kilo.

What is the expected cost per week if Frasse does not allow shortage? ...(2p)

(c) Now Frasse allows shortage. To keep his customers happy, if there is shortage he pays them 20 Euro per kilo for the coffee that they (honestly) claim that they would have demanded.

Assume that there are no ingoing inventory at the start of the period.
What is the optimal amount of coffee that Frasse should buy each week?
What is the optimal expected cost per week?
(Determine this as explicitly as possible)
Which strategy should Frasse use; To guarrantee no shortage or to pay off the customers who do not get their demanded coffee? .... (4p)

3. During his time as coffee management director at ESA Frasse has observed that the productivity and attitude of the personel at the agency has improved when serving coffee of fresh strong beans. From his experience, either the productivity and attitude is great or it is good. When it is great the revenue of the agency improves with 40 kEuro per week relative to when it is good. The increased cost of serving the fresh strong coffee is however 10 kEuro per week. Furthermore, Frasse has estimated the dynamics of the state of mind of the personel as described next. If the situation at the beginning of a week is great, the probability is 0.9 that the situation at the beginning of next week is great if he serves fresh strong coffee and 0.5 if he serves budget coffee. If it is good the probability is 0.9 that the situation at the beginning of next week is great if he serves fresh strong coffee and 0.5 if he serves budget coffee.

Frasse does not remember from his studies what the discount factor signifies. Knowing only that it should be between 0 and 1 he decides to let it be 0.5.

(a) The following optimization problem

$$(P) \quad \left[\begin{array}{ll} \text{minimize} & c^T y \\ \text{s.t.} & Ay = b \\ & y \in \mathbb{R}^4 \end{array}\right]$$

where

$$A = \left[\begin{array}{rrrr} 0.55 & 0.75 & -0.45 & -0.25 \\ -0.05 & -0.25 & 0.95 & 0.75 \end{array}\right], \quad b = \left[\begin{array}{c} 1/2 \\ 1/2 \end{array}\right],$$

and

$$c^T = \left[\begin{array}{cccc} -30 & -40 & 10 & 0 \end{array}\right],$$

has been solved and the optimal y is given by $\hat{y} = [0\ 1\ 0\ 1]^T$.
Show that this optimization problem should solve the optimal policy problem. What is the optimal policy ? and the optimal value ?
.... (3p)

(b) Frasse decides to check his results by using the policy improvement algorithm for solving this problem. Help him to do this, use the algorithm and verify that the optimal policy is correct and that the optimal values are consistent.
.... (4p)

(c) For this problem it would be more realistic to not use discounting. Determine if the problem without discounting has the same optimal policy as the problem with discounting above, *i.e.*, if Frasse is lucky enough to solve the real problem. Note that you do not need to determine the optimal policy. .... (4p)

4. Since the parking situation at the big office is difficult, Frasse gets the job to plan for two new huge parking lots, A and B that will be built. It is assumed that the (external) arrivals to the parking lots are independent Poisson processes with intensities 470 and 940 cars per hour for A and B respectively. The parking behaviour is assumed to be the same for all drivers, namely, they will drive around for an exponential time with mean value of 10 minutes before finding a parking spot or an exponential time with mean value of 15 minutes after which he gives up and leaves the parking lot, whichever occurs first. These times are all assumed independent of all other drivers random times. Among those who gives up and leaves parking lot A, with probability 1/2 they go to try their luck at parking lot B and the others gives up permanently and goes home. Among those who gives up and leaves parking lot B, with probability 3/4 they go to try their luck at parking lot A and the others gives up permanently and goes home.

Use a reasonable model to describe this parking system, and motivate well any further assumptions/approximations made.

(a) What is the fraction between the number of people that eventually finds a parking spot and those who leave for home? .... (3p)

(b) What is the average time from that a random car arrives to the parking lots to that they either find a parking spot or leaves for home? ....(6p)

(c) At some particular time instant, what is the average number of cars cruising around parking lot A trying to find an empty spot? ....(3p)

5. Frasse has been promoted to chief of a small research group. The group will perform three different experiments and depending on how the recruitment goes he will have between four and eight (equally talented and equally payed) researchers available for performing the experiments.

Frasse has estimated the probability (in per cent) that experiment i will fail if n researchers are assigned to the experiment to be given by the function $p_i(n)$ tabulated below.

n	$p_1(n)$	$p_2(n)$	$p_3(n)$
1	20	30	40
2	15	20	25
3	13	16	20
4	11	12	17
5	10	10	15

Assume that there is at least one resercher on every experiment.

(a) Is $p(n_1, n_2, n_3) = p_1(n_1) + p_2(n_2) + p_3(n_3)$ a separable integer convex function? (3p)

(b) Determine how many researchers should be on each project if Frasse aims at minimizing the sum of the risks of failure for the individual experiments for the the five cases of: 4,5,6,7,8 total researchers available.

If there are more than one optimal solution for some fixed number of total researchers, determine all such optimal solutions. ....(6p)

Good luck!