KTH Mathematics

Exam in SF2863 Systems Engineering and SF2862 Stochastic decision support models
Monday June 7, 2010, 14.00–19.00

Examiner: Krister Svanberg, tel. 790 71 37

Allowed tools: Pen/pencil, ruler and eraser. A formula-sheet is handed out.
No calculator is allowed!

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Explain carefully the methods you use, in particular if you use a method not taught in the course. Conclusions should always be motivated.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each sheet!
25 points, including bonus points from the homeworks, are sufficent for passing the exam. For 23-24 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam have been reported. Contact the examiner as soon as possible for such a completion.

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1. A queueing system consists of three service facilities, called A, B and C.
Customers arrive to the system according to a poisson process with rate 7 customers per hour. Each new customer first go to facility A.
Customers who have been served at A will either go to B, with probability 0.5, or immediatly leave the whole system, with probability 0.5.
Customers who have been served at B will either go to C, with probability 0.5, or immediatly leave the whole system, with probability 0.5.
Customers who have been served at C will either go to A, with probability 0.5, or immediatly leave the whole system, with probability 0.5.
In each of the facilities there is a single server with exponentially distributed service time with mean 5 minutes.

(a) Calculate the average number of customers in the whole system. .... (5p)

(b) Consider a customer which has just arrived to facility B. Calculate the expected remaining time until this customer leaves the whole system. ... (3p)

(c) Consider a customer which has just started being served in facility B. Calculate the expected remaining time until this customer leaves the whole system. .... (2p)

(d) Calculate the probability that (at a random point in time) there are equally many customers in all three facilities. .... (2p)

2. In a satellite, planned to be launched into space for a one year mission, a certain measurement system relies on n different types of instruments which all must work in order for the whole system to work. If an instrument fails, it can not be repaired during the mission. In order to increase the reliability of the system, the instruments can be duplicated, but then the weight of the satellite increases.

Assume that an instrument of type k fails with probability p_k during the mission, regardless of if it is in use or in stand-by, and that failures of instruments happen independent of each other.

If $x_k + 1$ instruments of type k is brought to the mission, where $x_k \in \{0, 1, 2, \ldots\}$, then the probability that at least one of the instruments of type k will be working during the whole mission is $1-p_k^{x_k+1}$, and then the probability that the measurement system will be working during the whole mission is given by the product

$$\prod_{k=1}^{n}(1 - p_k^{x_k+1}).$$

To get a separable function, we take the logarithm of this product, and we also make a sign change (due to our habit of prefering minimization to maximization) and consider the function

$$f(\mathbf{x}) = \sum_{k=1}^{n} -\log(1 - p_k^{x_k+1}).$$

The total weight of the instruments brought to the satellite is given by the function

$$g(\mathbf{x}) = \sum_{k=1}^{n} w_k(x_k + 1),$$

where w_k is the given weight of each instrument of type k.

One would like both of these functions $f(\mathbf{x})$ and $g(\mathbf{x})$ to be "small", but there is an obvious conflict here. Therefore, one should chose one of the *efficient solutions* corresponding to the pair (f, g).

Your task is to calculate all efficient solutions with $x_k \geq 0$ for all k and with a total weight $\leq W^{\max}$, for the following data:

$$n = 3, \quad W^{\max} = 15, \quad (w_1, w_2, w_3) = (3, 1, 2), \quad (p_1, p_2, p_3) = (0.3,\ 0.1,\ 0.2).$$

Since you are not allowed to use any calculator, you should use the approximation $\log(1-z) \approx -z$ for "small" z (obtained e.g. by a Taylor expansion) and thus consider the function

$$f(\mathbf{x}) = \sum_{k=1}^{n} p_k^{x_k+1},$$

instead of the function $f(\mathbf{x}) = \sum_{k=1}^{n} -\log(1 - p_k^{x_k+1})$. . . . . (8p)

3. A company has two machines, called M_1 and M_2, which work independently of each other, and which sometimes break down and must be repaired. The time to failure for M_i is exponentially distributed with expected value $1/\lambda_i$, for $i = 1, 2$, while the repair time for M_i is exponentially distributed with expected value $1/\mu_i$, for $i = 1, 2$. Here, λ_1, λ_2, μ_1 and μ_2 are given positive real numbers.

If a machine breaks down while the other machine is under repair, the repair of the latter machine must be completed before the repair of the former can be started. (The single repair man always completes a repair before starting a new one.)

Let P_b = the part of time both machines are broken,
and P_w = the part of time both machines are working.

The managment of the company wants to find out how these quantities P_b and P_w depend on the four numbers λ_1, λ_2, μ_1 and μ_2. You should help them to do that.

(a) First assume that the two machines are "identical", so that $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$. Then just three states are needed to analyze the system, and to answer the questions of how P_b and P_w depend on λ and μ.
Make this analysis, and answer the questions. .... (5p)

(b) Next, assume that the machines are not identical. Then more than three states are needed to analyze the system, and to answer the questions of how P_b and P_w depend on λ_1, λ_2, μ_1 and μ_2.
Formulate the system of equations which must be solved in order to find the answers the questions (it is not required that you solve the system), and explain carefully how the answers to the questions follow from the solution of your formulated system of equation. .... (4p)

(c) Finally, you should check the consistency of the system of equation you formulated in (b) with your results from (a) by considering a simple special case.
Use your work from (a) to suggest a solution to the equations in (b) for the special case that $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, and check that this suggested solution is indeed a solution to your equations in (b). .... (2p)

4. A certain *control system* is described by the following time discrete dynamic model:

$$x_{k+1} = x_k + u_k, \quad k = 0, 1, 2, \ldots$$

where x_k is the *state* of the system at time k, while u_k is the *control* at time k. Both x_k and u_k are real numbers.

At time $k = 0$, the system is in state $x_0 = 13$, and the task is to chose the controls u_0, u_1 and u_2 in such a way that the following cost function is minimized

$$\sum_{k=0}^{2} (x_k^2 + u_k^2) + x_3^2 \, .$$

This should be done by using Dynamic programming. Therefore, let:

$V_k(x_k)$ = the total remaining cost from time k (including the terms x_k^2 and u_k^2), given that the system is in state x_k at time k.
In particular, $V_3(x_3) = x_3^2$.

(a) Formulate a recursive equation for $V_k(x_k)$. .... (3 p)

(b) Use this recursive equation to calculate the optimal controls u_0, u_1 and u_2, the resulting states x_1, x_2 and x_3, and the total cost $V_0(13)$. .... (6 p)

5. A bakery produces some very odd sandwiches for a quite peculiar but rich customer. The customer buys the sandwiches on friday afternoon each week, but it is not known in advance how many of them the customer will demand. Based on some experience, the manager of the bakery has figured out that the number of sandwiches demanded by the customer a friday can be modelled as a stochastic variable ξ with distribution $P(\xi = k) = 0.5^k$, for $k = 1, 2, \ldots$, and $P(\xi = 0) = 0$ (the customer always shows up.)

The cost for the bakery of producing x sandwiches a friday morning can be approximated by $c(x) = 50 + 10x$ SEK, for $x = 1, 2, \ldots$, and $c(0) = 0$.

The customer pays 100 SEK for each sandwich.

No other customer (and none in the bakery staff) wants this kind of sandwich, and the sandwiches can not be stored (due to the smell), so produced sandwiches which are not sold to this customer are immediately thrown away.

The question you should help the bakery to answer is how many sandwiches the bakery should produce each friday morning in order to maximize their expected profit. This should be done for each of the following two cases:

(a) If the customer demands ξ sandwiches and the bakery has produced x, then the customer will buy ξ if $x \geq \xi$ and x if $x < \xi$. .... (4p)

(b) If the customer demands ξ sandwiches and the bakery has produced x, then the customer will buy ξ if $x \geq \xi$, but no sandwiches at all if $x < \xi$. (But he will anyhow return as usual next friday.) .... (6p)

Good luck!