# Solutions to the exam in SF2863, December 2009

**Exercise 1.**

Let $p(k) = P(X = k)$, where $X$ is the number of engines in the repair shop.

Since the repair shop is an $M/M/1$ system with $\lambda = 0.3$, $\mu = 0.5$ and $\rho = \lambda/\mu = 0.6$, we have the formulas

$$p(0) = 1-\rho = 0.4, \quad p(k) = \rho^k p(0) = 0.4 \cdot 0.6^{\,k} \text{ and } L = \mathrm{E}[X] = \frac{\rho}{1-\rho} = \frac{0.6}{0.4} = 1.5\,.$$

**1.(a)** $\mathrm{E}[X] = 1.5$, according to above.

**1.(b)** $P(X \geq 2) = p(2) + p(3) + \ldots = 1 - p(0) - p(1) = 1 - 0.4 - 0.24 = 0.36$.

**1.(c)** The number of grounded aircrafts is given by $(X-s)^+$, so the probability that there is at least one grounded aircraft is $P(X \geq s+1) = 1 - P(X \leq s) = 1 - (p(0) + \ldots + p(s))$.

$s = 0 \Rightarrow P(X \geq s+1) = 1 - p(0) = 0.6$.
$s = 1 \Rightarrow P(X \geq s+1) = 1 - p(0) - p(1) = 0.36$.
$s = 2 \Rightarrow P(X \geq s+1) = 1 - p(0) - p(1) - p(2) = 0.216$.

**1.(d)** Now we are searching for $\mathrm{E}[(X-s)^+]$, which we denote by $\mathrm{EBO}(s)$.

We have by definition that

$\mathrm{EBO}(s) = \mathrm{E}[(X-s)^+] = p(s+1) + 2p(s+2) + 3p(s+3) + \ldots$

and thus $\mathrm{EBO}(s+1) = p(s+2) + 2p(s+3) + 3p(s+4) + \ldots$

From these expressions, it follows that
$\mathrm{EBO}(s) - \mathrm{EBO}(s+1) = p(s+1) + p(s+2) + p(s+3) + \ldots = 1 - (p(0) + \ldots + p(s))$.

Thus, $\mathrm{EBO}(s+1) = \mathrm{EBO}(s) - R(s)$, where $R(s) = 1 - (p(0) + \ldots + p(s))$.

So we get that

$\mathrm{EBO}(0) = \mathrm{E}[(X-0)^+] = \mathrm{E}[X] = 1.5$.

$\mathrm{EBO}(1) = \mathrm{EBO}(0) - R(0) = 1.5 - (1 - 0.4) = 0.9$.

$\mathrm{EBO}(2) = \mathrm{EBO}(1) - R(1) = 0.9 - (1 - 0.4 - 0.24) = 0.54$.

**Exercise 2.**

The arrival rates to the two facilities are obtained from the system

$\lambda_A = 6 + 0.5\lambda_B$ and $\lambda_B = 0.8\lambda_A$, which gives that $\lambda_A = 10$ and $\lambda_B = 8$.

In the first situation, both $A$ and $B$ are $M/M/1$ with $\mu_A = 12$ and $\mu_B = 10$, so that $\rho_A = \lambda_A/\mu_A = 5/6 < 1$ and $\rho_B = \lambda_B/\mu_B = 4/5 < 1$.

The average number of customers in facility A becomes $L_A^{(1)} = \dfrac{\rho_A}{1-\rho_A} = 5$,
while the average number of customers in facility B becomes $L_B^{(1)} = \dfrac{\rho_B}{1-\rho_B} = 4$,
so that the average number of customers in the system becomes $L_A^{(1)} + L_B^{(1)} = 9$.

**2.(a)**
If facility $A$ is changed to $M/M/2$, then $\rho_A = \lambda_A/(2\mu_A) = 5/12 < 1$, and then the average number of customers in facility A becomes $L_A^{(2)} = \dfrac{2\rho_A}{1-\rho_A^2} = \dfrac{120}{119} \approx 1$,
so that the average number of customers in the system becomes $L_A^{(2)} + L_B^{(1)} \approx 5$.

If instead facility $B$ is changed to $M/M/2$, then $\rho_B = \lambda_B/(2\mu_B) = 4/10 < 1$, and then the average number of customers in facility B becomes $L_B^{(2)} = \dfrac{2\rho_B}{1-\rho_B^2} = \dfrac{20}{21} \approx 1$,
so that the average number of customers in the system becomes $L_A^{(1)} + L_B^{(2)} \approx 6$.

Thus, the optimal place for the third server is in facility A.

**2.(b)**
If there are two servers in each facility, then the average number of customers in the system becomes $L_A^{(2)} + L_B^{(2)} = \dfrac{120}{119} + \dfrac{20}{21} \approx 2$.

Since $L_A^{(3)} + L_B^{(1)} > L_B^{(1)} = 4$, and $L_A^{(1)} + L_B^{(3)} > L_A^{(1)} = 5$,

it is certainly better with 2+2 servers than with 3+1 or 1+3.

**2.(c)**
Let $W_A$ denote the expected remaining time in the system for a customer who comes to facility $A$, and let $W_B$ denote the expected remaining time in the system for a customer who comes to facility $B$. Then

$W_A = V_A + 0.8\,W_B$ and $W_B = V_B + 0.5\,W_A$, where $V_A = \dfrac{L_A}{\lambda_A}$ and $V_B = \dfrac{L_B}{\lambda_B}$,

which gives that $W_A = \dfrac{V_A + 0.8V_B}{0.6} = \dfrac{L_A + L_B}{6}$.

Since each new customer who arrives to the system first go to facility $A$, the expected time in the system for an arriving customer is precisely $W_A$.

But since $W_A = \frac{1}{6}(L_A + L_B)$, minimizing $W_A$ is equivalent to minimizing $L_A + L_B$!

Thus, the conclusions from (a) and (b) are unchanged.

**Exercise 3.**

We will apply the marginal allocation algorithm. First we identify the functions $f$ and $g$:

$$f(s) = \sum_{j=1}^{4} \frac{c_j}{s_j + 1} = \sum_{j=1}^{4} f_j(s_j), \quad g(s) = \sum_{j=1}^{4} s_j = \sum_{j=1}^{4} g_j(s_j),$$

where $f_j(s_j) = \dfrac{c_j}{s_j + 1}$ and $g_j(s_j) = s_j$.

Clearly, $f$ is a decreasing separable function. Since $c_j/(1+x)$ is a convex function for $x > 0$, $f$ is integer-convex. Further, $g$ is obviously an increasing integer-convex separable function. If the functions $f_j(s_j)$ are evaluated for some reasonable values, the following table is obtained:

| k | $f_1(k)$ | $f_2(k)$ | $f_3(k)$ | $f_4(k)$ |
|---|---|---|---|---|
| 0 | $\frac{18}{0+1} = 18$ | $\frac{30}{0+1} = 30$ | $\frac{48}{0+1} = 48$ | $\frac{66}{0+1} = 66$ |
| 1 | $\frac{18}{1+1} = 9$ | $\frac{30}{1+1} = 15$ | $\frac{48}{1+1} = 24$ | $\frac{66}{1+1} = 33$ |
| 2 | $\frac{18}{2+1} = 6$ | $\frac{30}{2+1} = 10$ | $\frac{48}{2+1} = 16$ | $\frac{66}{2+1} = 22$ |
| 3 | $\frac{18}{3+1} = 4.5$ | $\frac{30}{3+1} = 7.5$ | $\frac{48}{3+1} = 12$ | $\frac{66}{3+1} = 16.5$ |

Then it is easy to determine the marginal quotients $-\Delta f_j(k)/\Delta g_j(k) = -\Delta f_j(k)$:

| k | $-\Delta f_1(k)$ | $-\Delta f_2(k)$ | $-\Delta f_3(k)$ | $-\Delta f_4(k)$ |
|---|---|---|---|---|
| 0 | 9 | 15 | 24 | 33 |
| 1 | 3 | 5 | 8 | 11 |
| 2 | 1.5 | 2.5 | 4 | 5.5 |

We can order the elements in this table:

| k | $-\frac{\Delta f_1(k)}{1}$ | $-\frac{\Delta f_2(k)}{1}$ | $-\frac{\Delta f_3(k)}{1}$ | $-\frac{\Delta f_4(k)}{1}$ |
|---|---|---|---|---|
| 0 | $\boxed{5}$ | $\boxed{3}$ | $\boxed{2}$ | $\boxed{1}$ |
| 1 | | | $\boxed{6}$ | $\boxed{4}$ |
| 2 | | | | $\boxed{7}$ |

The marginal allocation algorithm starts with $s^{(0)} = (s_1^{(0)}, s_2^{(0)}, s_3^{(0)}, s_4^{(0)}) = (0,0,0,0)$, and the generated efficient points are

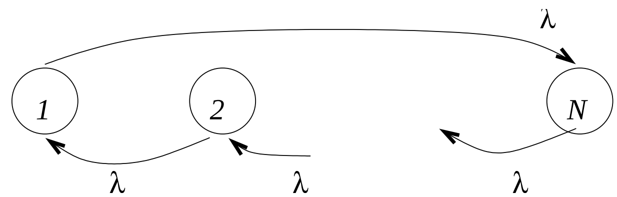
$s^{(1)} = (0,0,0,1)$,
$s^{(2)} = (0,0,1,1)$,
$s^{(3)} = (0,1,1,1)$,
$s^{(4)} = (0,1,1,2)$,
$s^{(5)} = (1,1,1,2)$,
$s^{(6)} = (1,1,2,2)$,
$s^{(7)} = (1,1,2,3)$.

Since $g(s^{(7)}) = 7$, it is well known from the theory of marginal allocation that the point $s^{(7)}$ is an optimal solution to the problem: minimize $f(s)$ subject to $g(s) \leq 7$.

The 7 additional consultants should thus be allocated as $1,1,2,3$ to the respective jobs, which means that the 11 consultants should be allocated as $2,2,3,4$.

**Exercise 4.**

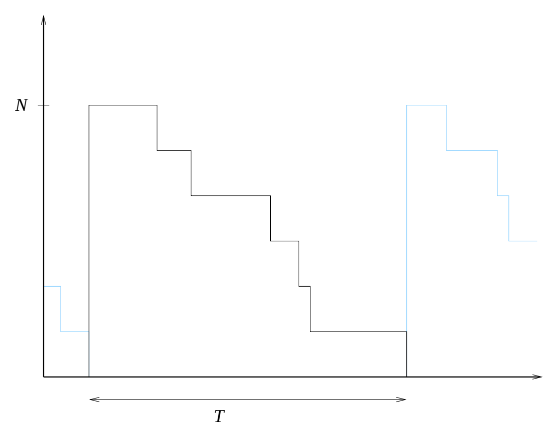
Since the inventory is immediately filled when it becomes empty, we get the following state diagram, where $\lambda = 5$.




The corresponding balance equations $\pi Q = 0$ (jumps out = jumps in) for obtaining the stationary distribution become

$\pi_1\lambda = \pi_2\lambda,\ \pi_2\lambda = \pi_3\lambda,\ \ldots\ ,\ \pi_{N-1}\lambda = \pi_N\lambda,\ \pi_N\lambda = \pi_1\lambda$, together with $\pi_1 + \cdots + \pi_N = 1$.

The unique solution of these equations is $\pi_j = 1/N$ for all $j = 1, \ldots, N$, and the average level of the inventory becomes $\displaystyle\sum_{j=1}^{N} j\,\pi_j = \frac{1}{N}\sum_{j=1}^{N} j = \frac{1}{N}\cdot\frac{N(N+1)}{2} = \frac{N+1}{2}$.




The expected number of jumps per day is $\lambda$. Each $N$:th jump corresponds to a replenishment of the inventory, so the expected number of replenishments per day is $\lambda/N$.

This gives the following natural objective function (average cost per day):

$C(N) = \dfrac{K\lambda}{N} + \dfrac{h(N+1)}{2}$, where $K = 1000$ and $h = 1$.

$N$ should be an integer, but we first ignore this and consider $N$ as a continuous variable. Then we can use calculus and obtain

$C'(N) = -\dfrac{K\lambda}{N^2} + \dfrac{h}{2}$, and $C''(N) = \dfrac{2K\lambda}{N^3} > 0$. Thus, $C(N)$ is strictly convex for all $N > 0$.

$C'(N) = 0$ gives $N^2 = \dfrac{2K\lambda}{h} = 10000$, so that $\hat{N} = 100$.

Since $\hat{N}$ is an integer, it is the optimal solution.

**Exercise 5.**

States: $G$ = Good, $B$ = Bad.
Decisions: $S$ = Standard overhaul, $E$ = Extended overhaul.
Transistion probabilities:
$p_{GG}(E) = 0.8$, $p_{GB}(E) = 0.2$, $p_{BG}(E) = 0.8$, $p_{BB}(E) = 0.2$,
$p_{GG}(S) = 0.6$, $p_{GB}(S) = 0.4$, $p_{BG}(S) = 0.6$, $p_{BB}(S) = 0.4$.

Expected immediate cost for different decisions in different states:
$C_{GS} = 1000 + p_{GB}(S) \cdot 5000 = 1000 + 0.4 \cdot 5000 = 3000$,
$C_{GE} = 3000 + p_{GB}(E) \cdot 5000 = 3000 + 0.2 \cdot 5000 = 4000$,
$C_{BS} = 10000 + p_{BB}(S) \cdot 5000 = 10000 + 0.4 \cdot 5000 = 12000$,
$C_{BE} = 14000 + p_{BB}(E) \cdot 5000 = 14000 + 0.2 \cdot 5000 = 15000$,

**(5.a):**
Let $V_i^{(n)}$ = the minimal expected remaining cost if the system is in state $i$ by the end of a week and there are $n$ more weeks to go. We get the recursive equations

$$V_G^{(n)} = \min\{\, C_{GS} + p_{GG}(S) V_G^{(n-1)} + p_{GB}(S) V_B^{(n-1)} \,,\, C_{GE} + p_{GG}(E) V_G^{(n-1)} + p_{GB}(E) V_B^{(n-1)} \,\},$$

$$V_B^{(n)} = \min\{\, C_{BS} + p_{BG}(S) V_G^{(n-1)} + p_{BB}(S) V_B^{(n-1)} \,,\, C_{BE} + p_{BG}(E) V_G^{(n-1)} + p_{BB}(E) V_B^{(n-1)} \,\},$$

with the boundary condition $V_G^{(0)} = V_B^{(0)} = 0$. This gives, for $n = 1$,

$V_G^{(1)} = \min\{\, C_{GS} \,,\, C_{GE} \,\} = \min\{\, 3000 \,,\, 4000 \,\} = 3000$,
$V_B^{(1)} = \min\{\, C_{BS} \,,\, C_{BE} \,\} = \min\{\, 12000 \,,\, 15000 \,\} = 12000$,
so the optimal decisions if only one week remains are $d_G = S$ and $d_B = S$.

Next, for $n = 2$, we get

$V_G^{(2)} = \min\{\, 3000 + 0.6 \cdot 3000 + 0.4 \cdot 12000 \,,\, 4000 + 0.8 \cdot 3000 + 0.2 \cdot 12000 \,\} =$
$= \min\{\, 9600 \,,\, 8800 \,\} = 8800$,

$V_B^{(2)} = \min\{\, 12000 + 0.6 \cdot 3000 + 0.4 \cdot 12000 \,,\, 15000 + 0.8 \cdot 3000 + 0.2 \cdot 12000 \,\} =$
$= \min\{\, 18600 \,,\, 19800 \,\} = 18600$,

so the optimal decisions if two week remains are $d_G = E$ and $d_B = S$,
which in words becomes: *Make Extended overall if the system is Good, and Standard overhaul if the system is Bad.*

**(5.b):**
We should start with the policy $d_G = S$ and $d_B = S$, so we must calculate the three numbers $g$, $v_G$ and $v_B$, corresponding to this policy, from the system
$v_B = 0$,
$g + v_G = C_{GS} + p_{GG}(S)\, v_G + p_{GB}(S)\, v_B$,
$g + v_B = C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B$,
which becomes
$v_B = 0$,
$g + v_G = 3000 + 0.6\, v_G + 0.4\, v_B$,
$g + v_B = 12000 + 0.6\, v_G + 0.4\, v_B$,

which can be simplified to
$v_B = 0$,
$g + 0.4\, v_G = 3000$,
$g - 0.6\, v_G = 12000$,
with the unique solution $g = 6600$, $v_G = -9000$, $v_B = 0$.

The next step in the algorithm is to check if
$g + v_G = \min\{\, C_{GS} + p_{GG}(S)\, v_G + p_{GB}(S)\, v_B \,,\; C_{GE} + p_{GG}(E)\, v_G + p_{GB}(E)\, v_B \,\}$, and
$g + v_B = \min\{\, C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B \,,\; C_{BE} + p_{BG}(E)\, v_G + p_{BB}(E)\, v_B \,\}$.

First, is $g + v_G = \min\{\, C_{GS} + p_{GG}(S)\, v_G + p_{GB}(S)\, v_B,\; C_{GE} + p_{GG}(E)\, v_G + p_{GB}(E)\, v_B \,\}$?
The left hand side is $6600 - 9000 = -2400$, while the right hand side is
$\min\{\, 3000 + 0.6 \cdot (-9000)\,,\; 4000 + 0.8 \cdot (-9000)\,\} = \min\{\, -2400\,, -3200\} = -3200$.
Thus, the decision $d_G = S$ should be changed to $d_G = E$.

Next, is $g + v_B = \min\{\, C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B \,,\; C_{BE} + p_{BG}(E)\, v_G + p_{BB}(E)\, v_B \,\}$?
The left hand side is $6600 + 0 = 6600$, while the right hand side is
$\min\{\, 12000 + 0.6 \cdot (-9000)\,,\; 15000 + 0.8 \cdot (-9000)\,\} = \min\{\, 6600\,, 7800\} = 6600$.
Thus, the decision $d_B = S$ should be kept.

Our current policy is now $d_G = E$ and $d_B = S$, so we must calculate the three numbers $g$, $v_G$ and $v_B$, corresponding to this policy, from the system
$v_B = 0$,
$g + v_G = C_{GE} + p_{GG}(E)\, v_G + p_{GB}(E)\, v_B$,
$g + v_B = C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B$,
which becomes
$v_B = 0$,
$g + v_G = 4000 + 0.8\, v_G + 0.2\, v_B$,
$g + v_B = 12000 + 0.6\, v_G + 0.4\, v_B$,
which can be simplified to
$v_B = 0$,
$g + 0.2\, v_G = 4000$,
$g - 0.6\, v_G = 12000$,
with the unique solution $g = 6000$, $v_G = -10000$, $v_B = 0$.

The next step in the algorithm is to check if
$g + v_G = \min\{\, C_{GS} + p_{GG}(S)\, v_G + p_{GB}(S)\, v_B \,,\; C_{GE} + p_{GG}(E)\, v_G + p_{GB}(E)\, v_B \,\}$, and
$g + v_B = \min\{\, C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B \,,\; C_{BE} + p_{BG}(E)\, v_G + p_{BB}(E)\, v_B \,\}$.

First, is $g + v_G = \min\{\, C_{GS} + p_{GG}(S)\, v_G + p_{GB}(S)\, v_B,\; C_{GE} + p_{GG}(E)\, v_G + p_{GB}(E)\, v_B \,\}$?
The left hand side is $6000 - 10000 = -4000$, while the right hand side is
$\min\{\, 3000 + 0.6 \cdot (-10000)\,,\; 4000 + 0.8 \cdot (-10000)\,\} = \min\{\, -3000\,, -4000\} = -4000$.
Thus, the decision $d_G = E$ should be kept.

Next, is $g + v_B = \min\{\, C_{BS} + p_{BG}(S)\, v_G + p_{BB}(S)\, v_B \,,\; C_{BE} + p_{BG}(E)\, v_G + p_{BB}(E)\, v_B \,\}$?
The left hand side is $6000 + 0 = 6000$, while the right hand side is
$\min\{\, 12000 + 0.6 \cdot (-10000)\,,\; 15000 + 0.8 \cdot (-10000)\,\} = \min\{\, 6000\,, 7000\} = 6000$.
Thus, the decision $d_B = S$ should be kept.

The conclusion is that the policy $d_G = E$ and $d_B = S$ is optimal.

**5.(c)**

The four possible long-run policies are
$R^a$, in which $d_G = S$ and $d_B = S$,
$R^b$, in which $d_G = S$ and $d_B = E$,
$R^c$, in which $d_G = E$ and $d_B = S$,
$R^d$, in which $d_G = E$ and $d_B = E$.

The transition matrices corresponding to these policies are

$$P^a = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}, \; P^b = \begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{bmatrix}, \; P^c = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}, \; P^d = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix}.$$

The rowvector $\pi = (\pi_G, \pi_B)$ with the stationary distribution for a strategy $R$ is obtained from the system $\pi = \pi P$, which can be written $\pi(I - P) = (0, 0)$ where $I$ is the unity matrix, together with $\pi_G + \pi_B = 1$.

First, we have that $I - P^a = \begin{bmatrix} 0.4 & -0.4 \\ -0.6 & 0.6 \end{bmatrix}$, so the equations $\pi^a(I - P^a) = (0, 0)$ become

$0.4\,\pi_G^a = 0.6\,\pi_B^a$, which together with $\pi_G^a + \pi_B^a = 1$ gives that $\pi^a = (\pi_G^a,\, \pi_B^a) = (0.6,\, 0.4)$.

Note that the two equations in $\pi^a(I - P^a) = (0, 0)$ are equivalent.

Next, we have that $I - P^b = \begin{bmatrix} 0.4 & -0.4 \\ -0.8 & 0.8 \end{bmatrix}$, so the equations $\pi^b(I - P^b) = (0, 0)$ become

$0.4\,\pi_G^b = 0.8\,\pi_B^b$, which together with $\pi_G^b + \pi_B^b = 1$ gives that $\pi^b = (\pi_G^b,\, \pi_B^b) = (2/3,\, 1/3)$.

Note that the two equations in $\pi^b(I - P^b) = (0, 0)$ are equivalent.

Next, we have that $I - P^c = \begin{bmatrix} 0.2 & -0.2 \\ -0.6 & 0.6 \end{bmatrix}$, so the equations $\pi^c(I - P^c) = (0, 0)$ become

$0.2\,\pi_G^c = 0.6\,\pi_B^c$, which together with $\pi_G^c + \pi_B^c = 1$ gives that $\pi^c = (\pi_G^c,\, \pi_B^c) = (0.75,\, 0.25)$.

Note that the two equations in $\pi^c(I - P^c) = (0, 0)$ are equivalent.

Finally, we have that $I - P^d = \begin{bmatrix} 0.2 & -0.2 \\ -0.8 & 0.8 \end{bmatrix}$, so the equations $\pi^d(I - P^d) = (0, 0)$ become

$0.2\,\pi_G^d = 0.8\,\pi_B^d$, which together with $\pi_G^d + \pi_B^d = 1$ gives that $\pi^d = (\pi_G^d,\, \pi_B^d) = (0.8,\, 0.2)$.

Note that the two equations in $\pi^d(I - P^d) = (0, 0)$ are equivalent.

The average cost per week for $R^a$ is $\pi_G^a C_{GS} + \pi_B^a C_{BS} = 0.6 \cdot 3000 + 0.4 \cdot 12000 = 6600$.
The average cost per week for $R^b$ is $\pi_G^b C_{GS} + \pi_B^b C_{BE} = (2/3) \cdot 3000 + (1/3) \cdot 15000 = 7000$.
The average cost per week for $R^c$ is $\pi_G^c C_{GE} + \pi_B^c C_{BS} = 0.75 \cdot 4000 + 0.25 \cdot 12000 = 6000$.
The average cost per week for $R^d$ is $\pi_G^d C_{GE} + \pi_B^d C_{BE} = 0.8 \cdot 4000 + 0.2 \cdot 15000 = 6200$.

Again, the conclusion is that the policy $R^c$ is optimal: Make Extended overall if the system is Good, and Standard overhaul if the system is Bad.