Solutions to the exam in SF2863, June 2010

Exercise 1.

The arrival rates to the facilities are obtained from the system

$\lambda_A = 7 + 0.5\lambda_C,\ \lambda_B = 0.5\lambda_A,\ \lambda_C = 0.5\lambda_B,$ which gives that $\lambda_A = 8\,,\ \lambda_B = 4\,,\ \lambda_C = 2\,.$

All three facilities are $M/M/1$, with service rates $\mu_A = 12\,,\ \mu_B = 12\,,\ \mu_C = 12\,,$

so that $\rho_A = \lambda_A/\mu_A = 8/12 < 1,\ \rho_B = \lambda_B/\mu_B = 4/12 < 1,\ \rho_C = \lambda_C/\mu_C = 2/12 < 1.$

1.(a) The average number of customers in the system becomes

$$L_A + L_B + L_C = \frac{\rho_A}{1-\rho_A} + \frac{\rho_B}{1-\rho_B} + \frac{\rho_C}{1-\rho_C} = 2 + 0.5 + 0.2 = 2.7.$$

1.(b) Let W_i denote the expected remaining time in the system for a customer who has just arrived to facility i, for $i = A, B, C$.

Then $W_A = V_A + 0.5\,W_B\,,\ \ W_B = V_B + 0.5\,W_C,\ \ W_C = V_C + 0.5\,W_A,$

where $V_A = \dfrac{L_A}{\lambda_A} = 1/4,\ \ V_B = \dfrac{L_B}{\lambda_B} = 1/8,\ \ V_C = \dfrac{L_C}{\lambda_C} = 1/10,$

which gives that $W_A = \dfrac{54}{140},\ \ W_B = \dfrac{38}{140},\ \ W_C = \dfrac{41}{140}.$

So the expected remaining time in the system for a customer who has just arrived to facility B is $W_B = \dfrac{38}{140}.$

1.(c) The expected remaining time in the system for a customer who has just started being served in facility B is $= \dfrac{1}{\mu_B} + 0.5\,W_C = \dfrac{1}{12} + \dfrac{41}{280} = \dfrac{193}{840}.$

1.(d) Let N_i be the number of customers at facility i (at a random point in time). Then $P(N_i = k) = (1-\rho_i)\rho_i^k$.

The probablity that there are equally number of customers in all three facilities is

$$\sum_{k=0}^{\infty} P(N_A = k,\ N_B = k,\ N_C = k) \ = \ \sum_{k=0}^{\infty} P(N_A = k)P(N_B = k)P(N_C = k) \ =$$

$$= (1-\rho_A)(1-\rho_B)(1-\rho_C)\sum_{k=0}^{\infty}(\rho_A\rho_B\rho_C)^k = \frac{(1-\rho_A)(1-\rho_B)(1-\rho_C)}{1-\rho_A\rho_B\rho_C} = \frac{5}{26}.$$

Exercise 2.

We will apply the marginal allocation algorithm for the functions $f(\mathbf{x}) = \sum_{j=1}^{3} f_j(x_j)$ and $g(\mathbf{x}) = \sum_{j=1}^{3} g_j(x_j)$, where $f_j(x_j) = p_j^{x_j+1}$ and $g_j(x_j) = w_j(x_j+1)$.

We have that $\Delta f_j(x_j) = f_j(x_j+1) - f_j(x_j) = p_j^{x_j+1}(p_j-1) < 0$ and $\Delta f_j(x_j+1) - \Delta f_j(x_j) = p_j^{x_j+1}(p_j-1)^2 > 0$, so that f_j is decreasing and integer-convex.

Further, $\Delta g_j(x_j) = g_j(x_j+1) - g_j(x_j) = w_j > 0$ and $\Delta g_j(x_j+1) - \Delta g_j(x_j) = 0$, so that g_j is increasing and integer-convex.

The given data imply that $\Delta g_1(k) = 3$, $\Delta g_2(k) = 1$, $\Delta g_3(k) = 2$, for all $k = 0, 1, 2, \ldots$

Further, the data imply that

k	$f_1(k)$	$f_2(k)$	$f_3(k)$
0	0.3	0.1	0.2
1	0.09	0.01	0.04
2	0.027	0.001	0.008
3	0.0081	0.0001	0.0016

which implies that

k	$-\Delta f_1(k)$	$-\Delta f_2(k)$	$-\Delta f_3(k)$
0	0.21	0.09	0.16
1	0.063	0.009	0.032
2	0.0189	0.0009	0.0064

so that

k	$-\Delta f_1(k)/\Delta g_1(k)$	$-\Delta f_2(k)/\Delta g_2(k)$	$-\Delta f_3(k)/\Delta g_3(k)$
0	0.07	0.09	0.08
1	0.021	0.009	0.016
2	0.0063	0.0009	0.0032

We can order the elements in this last table, with the largest element first, etc.

k	$-\Delta f_1(k)/\Delta g_1(k)$	$-\Delta f_2(k)/\Delta g_2(k)$	$-\Delta f_3(k)/\Delta g_3(k)$
0	$\boxed{3}$	$\boxed{1}$	$\boxed{2}$
1	$\boxed{4}$	$\boxed{6}$	$\boxed{5}$

The marginal allocation algorithm starts with $\mathbf{x}^{(0)} = (0,0,0)$,
and the generated efficient points and their weights become
$\mathbf{x}^{(0)} = (0,0,0)$, $g(\mathbf{x}^{(1)}) = 6$,
$\mathbf{x}^{(1)} = (0,1,0)$, $g(\mathbf{x}^{(1)}) = 7$,
$\mathbf{x}^{(2)} = (0,1,1)$, $g(\mathbf{x}^{(2)}) = 9$,
$\mathbf{x}^{(3)} = (1,1,1)$, $g(\mathbf{x}^{(3)}) = 12$,
$\mathbf{x}^{(4)} = (2,1,1)$, $g(\mathbf{x}^{(4)}) = 15$.

Exercise 3.

3.(a). When the two machines are identical, we need three states:

State 0 : Both machines are working.
State 1 : One machine is working.
State 2 : No machine is working.

The balance equations for the state probabilities P_j become

$$2\lambda P_0 = \mu P_1\,, \quad \lambda P_1 = \mu P_2\,, \quad P_0 + P_1 + P_2 = 1\,,$$

with the solution

$$P_0 = \frac{\mu^2}{\mu^2 + 2\mu\lambda + 2\lambda^2}\,, \quad P_1 = \frac{2\mu\lambda}{\mu^2 + 2\mu\lambda + 2\lambda^2}\,, \quad P_2 = \frac{2\lambda^2}{\mu^2 + 2\mu\lambda + 2\lambda^2}\,.$$

The quantities P_b and P_w, which were asked for, are given by $P_b = P_2$ and $P_w = P_0$.

3.(b). When the two machines are not identical, we need five states:

State 0 : Both machines are working.
State 1 : M_1 is under repair, M_2 is working.
State 2 : M_1 is working, M_2 is under repair.
State 3 : Both machines are broken, M_1 is under repair while M_2 is waiting for repair.
State 4 : Both machines are broken, M_2 is under repair while M_1 is waiting for repair.

The balance equations for the state probabilities π_j become

$(\lambda_1 + \lambda_2)\pi_0 = \mu_1\pi_1 + \mu_2\pi_2$,
$(\mu_1 + \lambda_2)\pi_1 = \lambda_1\pi_0 + \mu_2\pi_4$,
$(\lambda_1 + \mu_2)\pi_2 = \lambda_2\pi_0 + \mu_1\pi_3$,
$\mu_1\pi_3 = \lambda_2\pi_1$,
$\mu_2\pi_4 = \lambda_1\pi_2$,
$\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$.

The quantities P_b and P_w, which were asked for, are given by $P_b = \pi_3 + \pi_4$ and $P_w = \pi_0$.

3.(c). Now assume that $\mu_1 = \mu_2 = \mu = 1$ and $\lambda_1 = \lambda_2 = \lambda = 1$.

Then the soultion in (a) becomes $P_0 = 0.2$, $P_1 = 0.4$, $P_2 = 0.4$, so that $P_b = 0.4$ and $P_w = 0.2$.

The obvious guess for a solution in (b) is then obtained from
$\pi_0 = P_0 = 0.2$, $\pi_1 + \pi_2 = P_1 = 0.4$, $\pi_3 + \pi_4 = P_2 = 0.4$, $\pi_1 = \pi_2$ and $\pi_3 = \pi_4$,
which give that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = \pi_5 = 0.2$.

It is easy to see that this solution satisfies the balance equations.
Thus, again, $P_b = 0.2 + 0.2 = 0.4$ and $P_w = 0.2$.

Exercise 4.

The recursive equation become

$$V_k(x_k) = \min_{u_k}\{x_k^2 + u_k^2 + V_{k+1}(x_k + u_k)\}$$

with the boundary condition $V_3(x_3) = x_3^2$.

Application of this recursive equation for $k = 2$, $k = 1$ and $k = 0$ gives

$\hat{u}_2 = -\dfrac{x_2}{2}$, $V_2(x_2) = \dfrac{3}{2}x_2^2$, $\hat{u}_1 = -\dfrac{3x_1}{5}$, $V_1(x_1) = \dfrac{8}{5}x_1^2$, $\hat{u}_0 = -\dfrac{8x_1}{13}$, $V_0(x_0) = \dfrac{21}{13}x_0^2$.

Since $x_0 = 13$, we get

$x_0 = 13 \;\Rightarrow\; \hat{u}_0 = -8 \;\Rightarrow\; x_1 = 5 \;\Rightarrow\; \hat{u}_1 = -3 \;\Rightarrow\; x_2 = 2 \;\Rightarrow\; \hat{u}_2 = -1 \;\Rightarrow\; x_3 = 1.$

The total cost is $V_0(13) = \dfrac{21}{13}13^2 = 273.$

(Check: $x_0^2 + u_0^2 + x_1^2 + u_1^2 + x_2^2 + u_2^2 + x_3^2 = 13^2 + 8^2 + 5^2 + 3^2 + 2^2 + 1^2 + 1^2 = 273.$)

Exercise 5.

Let $r(x)$ denote the expected revenue, let $c(x)$ denote the cost,
and let $f(x) = r(x) - c(x)$ denote the expected profit (a given friday).

There are two cases, either $x = 0$ or $x \geq 1$.
If $x = 0$ then $r(x) = c(x) = f(x) = 0$, which will later be compared with the best choice among all $x \geq 1$.

From now on, we will consider the case $x \geq 1$, i.e. $x \in \{1, 2, 3, \ldots\}$.
Then the cost is given by $c(x) = 50 + 10x$.

5.(a) Here, the revenue is 100ξ if $\xi \leq x$ and $100x$ if $\xi > x$.

Therefore, the expected revenue becomes

$$r(x) = 100\left(\sum_{k=1}^{x} kP(\xi = k) + xP(\xi \geq x{+}1) \right).$$

Then $r(x{+}1) = 100\left(\sum_{k=1}^{x+1} kP(\xi = k) + (x{+}1)P(\xi \geq x{+}2) \right)$,

which can be written equivalently as

$$r(x{+}1) = 100\left(\sum_{k=1}^{x} kP(\xi = k) + (x{+}1)P(\xi \geq x{+}1) \right).$$

From this, it immediately follows that

$\Delta r(x) = r(x{+}1) - r(x) = 100P(\xi \geq x{+}1) = 100(1 - F_\xi(x))$,

where $F_\xi(x) = P(\xi \leq x)$ is the distribution function for ξ.

Since $\Delta c(x) = c(x{+}1) - c(x) = 10$ (when $x \geq 1$), we get that

$\Delta f(x) = f(x{+}1) - f(x) = \Delta r(x) - \Delta c(x) = 100(1 - F_\xi(x)) - 10$.

This gives us the following equivalences:

"$x{+}1$ is better than x" $\Leftrightarrow$ $\Delta f(x) > 0$ $\Leftrightarrow$ $F_\xi(x) < \dfrac{90}{100}$.

Since $P(\xi = 0) = 0$ and $P(\xi = k) = \dfrac{1}{2^k}$ for $k \geq 1$, the distribution function becomes as follows for $x \geq 1$: $F_\xi(x) = \displaystyle\sum_{k=1}^{x} \frac{1}{2^k} = 1 - \frac{1}{2^x}$, and thus

"$x{+}1$ is better than x" $\Leftrightarrow$ $1 - \dfrac{1}{2^x} < \dfrac{90}{100}$ $\Leftrightarrow$ $2^x < 10$ $\Leftrightarrow$ $x \leq 3$.

From this we get that $f(1) < f(2) < f(3) < f(4) > f(5) > f(6) > \cdots$

so it follow that $x = 4$ is the best choice among all $x \geq 1$.

The formulas above give that $f(4) = r(4) - c(4) = 187.5 - 90 = 97.5$,
which is better than $f(0) = 0$.

Thus, $x = 4$ is the best choice among all $x \geq 0$.

5.(b) Now, the revenue is 100ξ if $\xi \le x$ and 0 if $\xi > x$.

Therefore, the expected revenue becomes $r(x) = 100 \sum_{k=1}^{x} kP(\xi = k)$.

Then $r(x+1) = 100 \sum_{k=1}^{x+1} kP(\xi = k)$, from which it follows that

$\Delta r(x) = r(x+1) - r(x) = 100(x+1)P(\xi = x+1) = \dfrac{100(x+1)}{2^{x+1}}$.

Since $\Delta c(x) = c(x+1) - c(x) = 10$ (when $x \ge 1$), we get that

$\Delta f(x) = f(x+1) - f(x) = \Delta r(x) - \Delta c(x) = \dfrac{100(x+1)}{2^{x+1}} - 10$.

This gives us the following equivalences:

"$x+1$ is better than x" $\Leftrightarrow$ $\Delta f(x) > 0$ $\Leftrightarrow$ $\dfrac{x+1}{2^{x+1}} > \dfrac{1}{10}$.

For $x = 1, 2, \ldots$, the left hand side in the last inequality becomes:

$\frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \frac{7}{128}, \ldots,$

from which it follows that $\Delta f(x) > 0$ for $x = 1, 2, 3, 4$, while $\Delta f(x) < 0$ for $x \ge 5$.

This implies that $f(1) < f(2) < f(3) < f(4) < f(5) > f(6) > f(7) > \cdots$

so it follow that $x = 5$ is the best choice among all $x \ge 1$.

The formulas above give that $f(5) = r(5) - c(5) = 178.125 - 100 = 78.125$,
which is better than $f(0) = 0$.

Thus, $x = 5$ is the best choice among all $x \ge 0$.