On marginal allocation

Krister Svanberg
Optimization and Systems Theory,
KTH, Stockholm, Sweden.
krille@math.kth.se

1. Minimizing an integer-convex function of a single variable

Let $\mathcal{N}$ denote the set of natural numbers (non-negative integers), $\mathcal{N} = \{0, 1, 2, \dots, \}$, and let $I\!R$ denote the set of real numbers.

Further, let f be a given function from $\mathcal{N}$ to $I\!R$, and consider the optimization problem

$$\text{minimize} \quad f(x) \quad \text{subject to } x \in \mathcal{N}\,. \tag{1.1}$$

Def: The number $\hat{x} \in \mathcal{N}$ is an *optimal solution* to (1.1) if $f(\hat{x}) \leq f(x)$ for all $x \in \mathcal{N}$.
In this case, the *optimal value* of the problem (1.1) is given by $f(\hat{x})$.

For a completely general function f, (1.1) might be an impossible problem to solve (since there is an infinite number of numbers to compare.) But if f has certain properties, (1.1) could be solvable, perhaps even easily solvable. One such property will be discussed next.

For each number $x \in \mathcal{N}$, let

$$\Delta f(x) = f(x{+}1) - f(x). \tag{1.2}$$

Def: The function f from $\mathcal{N}$ to $I\!R$ is *integer-convex* if $\Delta f(x{+}1) \geq \Delta f(x)$ for all $x \in \mathcal{N}$.

Prop 1.1: Assume that f is an integer-convex function from $\mathcal{N}$ to $I\!R$.
Then the number $\hat{x}$ is an optimal solution to problem (1.1) if and only if the following inequalities are satisfied:

$$\begin{aligned} \Delta f(\hat{x}{-}1) \leq \; 0 \leq \Delta f(\hat{x}) \quad & \text{if } \; \hat{x} > 0\,, \\ 0 \leq \Delta f(0) \quad & \text{if } \; \hat{x} = 0\,. \end{aligned} \tag{1.3}$$

Proof:
If $\hat{x} > 0$ and $\Delta f(\hat{x}{-}1) > 0$ then $f(\hat{x}{-}1) < f(\hat{x})$ and $\hat{x}$ is not optimal.
If $\hat{x} \geq 0$ and $\Delta f(\hat{x}) < 0$ then $f(\hat{x}{+}1) < f(\hat{x})$ and $\hat{x}$ is not optimal.
If $\hat{x} = 0$ and $\Delta f(0) \geq 0$ then, since f is integer-convex, $\Delta f(x) \geq 0$ for all $x \geq 0$.
This implies that $f(0) \leq f(1) \leq f(2) \leq \dots$, so that $\hat{x} = 0$ is optimal.
If $\hat{x} > 0$, $\Delta f(\hat{x}) \geq 0$ and $\Delta f(\hat{x}{-}1) \leq 0$ then, since f is integer-convex,
$\Delta f(x) \geq 0$ for all $x \geq \hat{x}$ and $\Delta f(x) \leq 0$ for all $x \leq \hat{x} - 1$. This implies that
$f(\hat{x}) \leq f(\hat{x}{+}1) \leq f(\hat{x}{+}2) \leq \dots$, and $f(\hat{x}) \leq f(\hat{x}{-}1) \leq \dots \leq f(0)$,
so that $\hat{x}$ is optimal.

These optimality criteria can obviously be used for solving problem (1.1):
If $\Delta f(x) < 0$ then x is too small to be optimal, while if $\Delta f(x{-}1) > 0$ then x is too large.

2. Minimizing integer-convex separable functions of several variables

Let $\mathcal{N}^n$ denote the set of vectors $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$ with natural numbers as components. Further, let f be a given function from $\mathcal{N}^n$ to $I\!R$, and consider the optimization problem

$$\text{minimize} \;\; f(\mathbf{x}) \;\; \text{subject to } \mathbf{x} \in \mathcal{N}^n \, . \tag{2.1}$$

Def: The vector $\hat{\mathbf{x}} \in \mathcal{N}^n$ is an *optimal solution* to (2.1) if $f(\hat{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{N}^n$. In this case, the *optimal value* of the problem (2.1) is given by $f(\hat{\mathbf{x}})$.

Def: The function f from $\mathcal{N}^n$ to $I\!R$ is *separable* if f can be written

$$f(\mathbf{x}) = \sum_{j=1}^{n} f_j(x_j), \tag{2.2}$$

where, for each $j = 1, \ldots, n$, f_j is a function from $\mathcal{N}$ to $I\!R$.

For separable functions, there is a natural definition of integer-convexity:

Def: The separable function f in (2.2) is *integer-convex* if each f_j is integer-convex.

Prop 2.1: Assume that f is an integer-convex separable function from $\mathcal{N}^n$ to $I\!R$. Then the vector $\hat{\mathbf{x}}$ is an optimal solution to problem (2.1) if and only if the following inequalities are satisfied for each $j = 1, \ldots, n$:

$$\begin{aligned} \Delta f_j(\hat{x}_j - 1) \leq \; 0 \; &\leq \Delta f_j(\hat{x}_j) \quad && \text{if } \; \hat{x}_j > 0 \, , \\ 0 \; &\leq \Delta f_j(0) \quad && \text{if } \; \hat{x}_j = 0 \, . \end{aligned} \tag{2.3}$$

Proof: The statement follows from Prop 1.1, together with the observation that the sum $f(\mathbf{x}) = \sum_j f_j(x_j)$ is minimized if and only if each term $f_j(x_j)$ is minimized (since there is no coupling between the variables).

3. Efficient points for two integer-convex separable functions

Let f and g be two given integer-convex separable functions from $\mathcal{N}^n$ to $I\!R$:

$$f(\mathbf{x}) = \sum_{j=1}^{n} f_j(x_j) \;\; \text{and} \;\; g(\mathbf{x}) = \sum_{j=1}^{n} g_j(x_j). \tag{3.1}$$

Further assume that both $f(\mathbf{x})$ and $g(\mathbf{x})$ stands for quantities that we would like to be small, but that each $f_j(x_j)$ is strictly decreasing in x_j while each $g_j(x_j)$ is strictly increasing in x_j. This causes a conflict: Large values on the variables x_j will tend to make $f(\mathbf{x})$ small, which is desirable, but $g(\mathbf{x})$ large, which is undesirable. Small values on the variables x_j will tend to make $g(\mathbf{x})$ small, which is desirable, but $f(\mathbf{x})$ large, which is undesirable. This section deals with how to compromise between these conflicting goals.

To summarize the assumptions:

$$\begin{aligned} \Delta f_j(x_j) \le \Delta f_j(x_j{+}1) < 0 \quad & \text{for all j and all } x_j \in \mathcal{N}, \\ 0 < \Delta g_j(x_j) \le \Delta g_j(x_j{+}1) \quad & \text{for all j and all } x_j \in \mathcal{N}. \end{aligned} \tag{3.2}$$

Let $g^{\max}$ be a given upper bound on the acceptable values of $g(\mathbf{x})$: Points $\mathbf{x}$ with $g(\mathbf{x}) > g^{\max}$ are assumed to be unacceptable (e.g. too expensive). Further let

$$X = \{\, \mathbf{x} \in \mathcal{N}^n \mid g(\mathbf{x}) \le g^{\max} \}. \tag{3.3}$$

Since each function g_j is strictly increasing and integer-convex, the set X contains a finite number of vectors $\mathbf{x}$, but this finite number may in practical applications be extremely large.

Def: The vector $\hat{\mathbf{x}} \in X$ is an *efficient solution* corresponding to the above setting if there are constants $\alpha > 0$ and $\beta > 0$ such that $\hat{\mathbf{x}}$ is an optimal solution to the following optimization problem in $\mathbf{x}$:

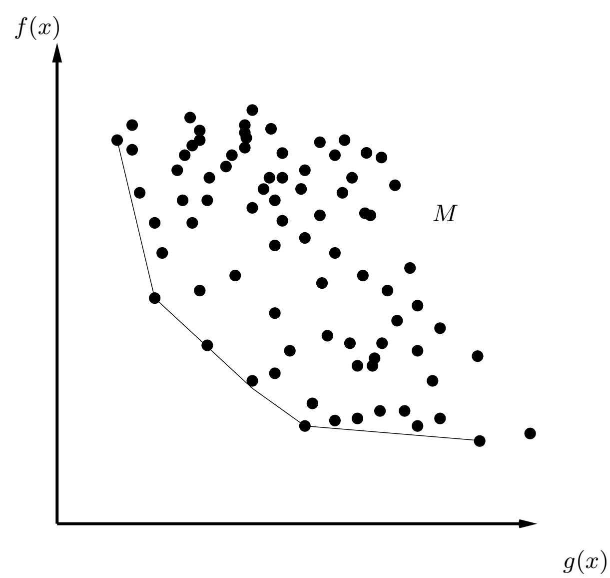
$$\text{minimize} \;\; \alpha\, g(\mathbf{x}) + \beta f(\mathbf{x}) \;\; \text{subject to } \mathbf{x} \in X\,. \tag{3.4}$$

Next, we will give a natural geometric interpretation of the efficient solutions defined above.

Let

$$M = \{\, (g(\mathbf{x}), f(\mathbf{x})) \mid \mathbf{x} \in X\} \subset I\!R^2. \tag{3.5}$$

This set M contains a finite (but possibly extremely large) number of points in $I\!R^2$. To get a picture of M, we may imagine that the points in M are plotted in a coordinate system where the horizontal axis shows $g(\mathbf{x})$ and the vertical axis shows $f(\mathbf{x})$.



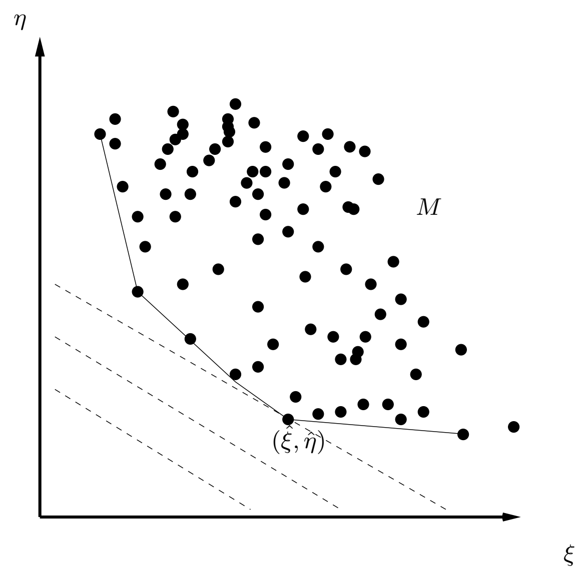
The *convex hull* of M is defined as the smallest *convex* set in $I\!R^2$ which contains M. Geometrically, the convex hull of M is what you get if you “stretch a rope” around M.

The *efficient curve* for the current setting is the piecewise linear curve that constitutes the "southwestern boundary" of the convex hull of M. Points $(g(\mathbf{x}), f(\mathbf{x})) \in M$ which lie on this efficient curve are called *efficient points*, and the corresponding vectors $\mathbf{x}$ are in fact the *efficient solutions* defined above. Here is an argument to motivate this last statement:

From a two-dimensional figure where the points of M are plotted and the convex hull of M is drawn, it follows that a point $(\hat{\xi}, \hat{\eta}) = (g(\hat{\mathbf{x}}), f(\hat{\mathbf{x}})) \in M$ belongs to the "southwestern boundary" of the convex hull of M if and only if there are constants $\alpha > 0$ and $\beta > 0$ such that $(\hat{\xi}, \hat{\eta})$ is an optimal solution to the following optimization problem in ξ and η:

$$\text{minimize } \ \alpha\,\xi + \beta\,\eta \ \text{ subject to } (\xi, \eta) \in M. \tag{3.6}$$

But this problem (3.6) is equivalent to the the above problem (3.4) in $\mathbf{x}$.



Typically, we are interested in determining the efficient curve for a given situation, with given functions f and g. It turns out that even if the number of points in M is extremely large, it is surprisingly easy to determine the efficient curve! We will describe below how this is done, but first some preparatory results.

Prop 3.1: The vector $\hat{\mathbf{x}} \in X$ minimizes $\alpha\, g(\mathbf{x}) + \beta f(\mathbf{x})$ subject to $\mathbf{x} \in X$ if and only if the following conditions are satisfied for each $j = 1, \ldots, n$:

$$\frac{-\Delta f_j(\hat{x}_j)}{\Delta g_j(\hat{x}_j)} \le \frac{\alpha}{\beta} \le \frac{-\Delta f_j(\hat{x}_j - 1)}{\Delta g_j(\hat{x}_j - 1)} \quad \text{if } \ \hat{x}_j > 0\,, \tag{3.7}$$

$$\frac{-\Delta f_j(0)}{\Delta g_j(0)} \le \frac{\alpha}{\beta} \quad \text{if } \ \hat{x}_j = 0\,, \tag{3.8}$$

Proof: Just replace $f(\mathbf{x})$ by $\alpha\, g(\mathbf{x}) + \beta f(\mathbf{x})$ in Prop 2.1

From this proposition, together with the above definition of an efficient solution, we get the following criteria for deciding whether a vector $\hat{\mathbf{x}}$ is an efficient solution or not:

Prop 3.2: $\hat{\mathbf{x}} \in X$ is an efficient solution if and only if there are constants $\alpha > 0$ and $\beta > 0$ such that the conditions (3.7)–(3.8) are satisfied for each $j = 1, \ldots, n$.

The following result shows that each efficient solution is in fact an optimal solution to two particular optimization problems.

Prop 3.3: Assume that $\hat{\mathbf{x}} \in X$ is an efficient solution, and let $\hat{g} = g(\hat{\mathbf{x}})$ and $\hat{f} = f(\hat{\mathbf{x}})$. Then $\hat{\mathbf{x}}$ is an optimal solution to both the following optimization problems:

$$\text{minimize} \;\; f(\mathbf{x}) \;\; \text{subject to} \;\; g(\mathbf{x}) \leq \hat{g}\,, \;\; \mathbf{x} \in X. \tag{3.9}$$

$$\text{minimize} \;\; g(\mathbf{x}) \;\; \text{subject to} \;\; f(\mathbf{x}) \leq \hat{f}\,, \;\; \mathbf{x} \in X. \tag{3.10}$$

Proof: If $\hat{\mathbf{x}}$ is an efficient solution then there are constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\, g(\hat{\mathbf{x}}) + \beta f(\hat{\mathbf{x}}) \leq \alpha\, g(\mathbf{x}) + \beta f(\mathbf{x}), \quad \text{for all } \mathbf{x} \in X. \tag{3.11}$$

First, take an arbitrary $\mathbf{x} \in X$ such that $g(\mathbf{x}) \leq \hat{g}$. Then, according to (3.11), it holds that

$$f(\hat{\mathbf{x}}) - f(\mathbf{x}) \leq (\alpha/\beta)(g(\mathbf{x}) - g(\hat{\mathbf{x}})) = (\alpha/\beta)(g(\mathbf{x}) - \hat{g}) \leq 0, \tag{3.12}$$

which implies that $\hat{\mathbf{x}}$ is an optimal solution to (3.9).

Next, take an arbitrary $\mathbf{x} \in X$ such that $f(\mathbf{x}) \leq \hat{f}$. Then, according to (3.11), it holds that

$$g(\hat{\mathbf{x}}) - g(\mathbf{x}) \leq (\beta/\alpha)(f(\mathbf{x}) - f(\hat{\mathbf{x}})) = (\beta/\alpha)(f(\mathbf{x}) - \hat{f}) \leq 0, \tag{3.13}$$

which implies that $\hat{\mathbf{x}}$ is an optimal solution to (3.10).

Important note:
Since the constants α and β are assumed to be > 0, and since $\hat{\mathbf{x}}$ minimizes $\alpha\, g(\mathbf{x}) + \beta f(\mathbf{x})$ if and only if $\hat{\mathbf{x}}$ minimizes $g(\mathbf{x}) + (\beta/\alpha) f(\mathbf{x})$, (and if and only if $\hat{\mathbf{x}}$ minimizes $(\alpha/\beta)\, g(\mathbf{x}) + f(\mathbf{x})$), we may without loss of generality assume that $\alpha = 1$ everywhere above (or, alternatively, that $\beta = 1$ everywhere above). This is sometimes done in applications of this theory.

4. Marginal allocation algorithm for generating efficient solutions

We will now describe a surprisingly simple algorithm for determining the efficient curve described above, but first we repeat the assumptions that f is integer-convex and strictly decreasing in each variable, while g is integer-convex and strictly increasing in each variable.

The algorithm start from $\mathbf{x}^{(0)} = \mathbf{0}$ (which is an efficient solution) and generates efficient solutions $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}, \ldots$ "from left to right", which means that each new generated point has a higher value on $g(\mathbf{x})$ but a lower value on $f(\mathbf{x})$ than the previously generated point.
Throughout the algorithm $\mathbf{x}^{(k)}$ denotes the k:th generated efficient solution.
The algorithm terminates when there is no longer any efficient solution with $g(\mathbf{x}) \le g^{\max}$.

Step 0:
Generate a table with n columns as follows. For $j = 1, \ldots, n$, fill the j:th column from the top and down with the quotients $-\Delta f_j(0)/\Delta g_j(0)$, $-\Delta f_j(1)/\Delta g_j(1)$, $-\Delta f_j(2)/\Delta g_j(2)$, etc... (A moderate number of quotients will suffice, additional quotients can be calculated as needed.) Note that the quotients are positive and strictly decreasing in each column.
Set $k = 0$, $\mathbf{x}^{(0)} = (0, \ldots, 0)^{\mathsf{T}}$, $g(\mathbf{x}^{(0)}) = g(\mathbf{0})$ and $f(\mathbf{x}^{(0)}) = f(\mathbf{0})$.
Let all the quotients in the table be *uncanceled*.

Step 1:
Select the *largest uncanceled* quotient in the table (if there are several equally large, choose one of these arbitrarily). *Cancel* this quotient and let ℓ be the number of the column from which the quotient was canceled.

Step 2:
Let $k := k + 1$. Then let $x_\ell^{(k)} = x_\ell^{(k-1)} + 1$ and $x_j^{(k)} = x_j^{(k-1)}$ for all $j \neq \ell$.
Further, calculate $f(\mathbf{x}^{(k)}) = f(\mathbf{x}^{(k-1)}) + \Delta f_\ell(x_\ell^{(k-1)})$ and $g(\mathbf{x}^{(k)}) = g(\mathbf{x}^{(k-1)}) + \Delta g_\ell(x_\ell^{(k-1)})$.
If $g(\mathbf{x}^{(k)}) \ge g^{\max}$, terminate the algorithm. Otherwise, go to Step 1.

Note that each generated solution $\mathbf{x}^{(k)}$ differs from the previously generated solution $\mathbf{x}^{(k-1)}$ in just one component. The name of the algorithm stems from the fact that

$$\frac{-\Delta f_j(x_j)}{\Delta g_j(x_j)} = \frac{\text{decrease in } f(\mathbf{x}) \text{ if } x_j \text{ is increased by 1}}{\text{increase in } g(\mathbf{x}) \text{ if } x_j \text{ is increased by 1}}.$$

Hence, in each step of the algorithm, we increase the x_j which gives marginally the largest decrease in $f(\mathbf{x})$ per increase in $g(\mathbf{x})$.

Prop 4.1: Each generated solution $\mathbf{x}^{(k)}$ is an efficient solution.

Proof: Consider a given generated solution $\mathbf{x}^{(k)}$. Choose $\alpha > 0$ and $\beta > 0$ such that $\alpha/\beta =$ the quotient that was canceled in Step 1 immediately before $\mathbf{x}^{(k)}$ was generated in Step 2. Then all canceled quotients are $\ge \alpha/\beta$, while all uncanceled quotients are $\le \alpha/\beta$. But then $\mathbf{x}^{(k)}$ and α/β satisfy conditions (3.7)–(3.8) for each $j = 1, \ldots, n$, which implies that $\mathbf{x}^{(k)}$ is an efficient solution.

Prop 4.2: Assume that all quotients $-\Delta f_j(x_j)/\Delta g_j(x_j)$ in the original table are different. Then the algorithm generates all efficient solutions which satisfy $g(\mathbf{x}) \le g^{\max}$.

Proof: Assume that $\hat{\mathbf{x}}$ is an efficient solution. Then the conditions (3.7)–(3.8) are satisfied for some α/β. But if all quotients are different, then α/β can be perturbed such that all inequalities in (3.7)–(3.8) becomes strict inequalities, so that, for each $j = 1, \ldots, n$,

$$\frac{-\Delta f_j(x_j)}{\Delta g_j(x_j)} < \frac{\alpha}{\beta} \quad \text{if } x_j \ge \hat{x}_j \,, \tag{4.1}$$

$$\frac{-\Delta f_j(x_j)}{\Delta g_j(x_j)} > \frac{\alpha}{\beta} \quad \text{if } x_j < \hat{x}_j \,. \tag{4.2}$$

These conditions determine $\hat{\mathbf{x}}$ uniquely. However, this solution will actually be generated by the algorithm in the stage where the latest canceled quotient is $> \alpha/\beta$, while the largest quotient that has not yet been canceled is $< \alpha/\beta$.

5. An important special case

Assume that $g_j(x_j) = cx_j$, where c is a positive constant which do not depend on j, so that

$$g(\mathbf{x}) = c \sum_{j=1}^{n} x_j \,. \tag{5.1}$$

Then the generated efficient points satisfy $g(\mathbf{x}^{(k)}) = c\,k$, for $k = 1, 2, \ldots$

It then follows from Prop 3.3 that, for each $k \in \mathcal{N}$, $\mathbf{x}^{(k)}$ is an optimal solution to the problem

$$\text{minimize} \;\; f(\mathbf{x}) \;\; \text{subject to} \;\; g(\mathbf{x}) \le c\,k \,, \;\; \mathbf{x} \in X. \tag{5.2}$$

Moreover, since $g(\mathbf{x})/c = \sum_j x_j \in \mathcal{N}$ for *all* $\mathbf{x} \in X$, it follows that if the constant b_k satisfies $c\,k \le b_k < c\,(k{+}1)$, then $\mathbf{x}^{(k)}$ is an optimal solution also to the problem

$$\text{minimize} \;\; f(\mathbf{x}) \;\; \text{subject to} \;\; g(\mathbf{x}) \le b_k \,, \;\; \mathbf{x} \in X. \tag{5.3}$$

Thus, for any right hand side $b > 0$, we can solve the problem

$$\text{minimize} \;\; f(\mathbf{x}) \;\; \text{subject to} \;\; g(\mathbf{x}) \le b \,, \;\; \mathbf{x} \in X. \tag{5.4}$$

Just let k be obtained by rounding b/c downwards to the nearest integer. Then $\mathbf{x}^{(k)}$ is an optimal solution.