Exercise 1.10

(1) $\sup\,(0,1] = 1$. Indeed, 1 is an upper bound, and moreover, if u is also an upper bound, then $1 \le u$ (since $1 \in S$). $\sup\,(0,1] = 1 \in (0,1]$, and so $\max\,(0,1] = \sup\,(0,1] = 1$.

$\inf\,(0,1] = 0$. First of all, 0 is a lower bound. Let l be a lower bound of $(0,1]$. We prove that $l \le 0$. (We do this by supposing that $l > 0$, and arriving at a contradiction. The contradiction is obtained as follows: if $l > 0$, then we will see that the average of 0 and l, namely $\frac{l}{2}$, is an element in $(0,1]$ that is less than the lower bound l, which is a contradiction.) If $l > 0$, then $0 < \frac{l}{2}$. Moreover, since $l \le 1$ (l is a lower bound of $(0,1]$ and $1 \in S$) it follows that $\frac{l}{2} \le \frac{1}{2} \le 1$. Thus $\frac{l}{2} \in (0,1]$. But since $l > 0$, it follows that $\frac{l}{2} < l$, a contradiction. Hence $l \le 0$.

Since $\inf\,(0,1] = 0 \notin (0,1]$, $\min\,(0,1]$ does not exist.

(For the other parts, we will not give these detailed arguments.)

(2) $\sup\,[0,1] = 1 \in [0,1]$, and so $\max\,[0,1] = \sup\,[0,1] = 1$.
$\inf\,[0,1] = 0 \in [0,1]$, and so $\min\,[0,1] = \inf\,[0,1] = 0$.

(3) $\sup\,(0,1) = 1 \notin (0,1)$ and so $\max\,(0,1)$ does not exist.
$\inf\,(0,1) = 0 \notin (0,1)$ and so $\min\,(0,1)$ does not exist.

(4) $\sup\{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\} = 1 \in \{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\}$,
and so $\max\{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\} = 1$.
$\inf\{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\} = -1 \in \{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\}$,
and so $\min\{\frac{1}{n} : n \in \mathbb{Z}\setminus\{0\}\} = -1$.

(5) $\sup\{-\frac{1}{n} : n \in \mathbb{N}\} = 0 \notin \{-\frac{1}{n} : n \in \mathbb{N}\}$ and so
$\max\{-\frac{1}{n} : n \in \mathbb{N}\}$ does not exist.
$\inf\{-\frac{1}{n} : n \in \mathbb{N}\} = -1 \in \{-\frac{1}{n} : n \in \mathbb{N}\}$ and so
$\min\{-\frac{1}{n} : n \in \mathbb{N}\} = -1$.

(6) $\sup\{\frac{n}{n+1} : n \in \mathbb{N}\} = 1 \notin \{\frac{n}{n+1} : n \in \mathbb{N}\}$ and so
$\max\{\frac{n}{n+1} : n \in \mathbb{N}\}$ does not exist.
$\inf\{\frac{n}{n+1} : n \in \mathbb{N}\} = \frac{1}{2} \in \{\frac{n}{n+1} : n \in \mathbb{N}\}$ and so
$\min\{\frac{n}{n+1} : n \in \mathbb{N}\} = \frac{1}{2}$.

(7) $\sup\{x \in \mathbb{R} : x^2 \le 2\} = \sqrt{2} \in \{x \in \mathbb{R} : x^2 \le 2\}$ and so
$\max\{x \in \mathbb{R} : x^2 \le 2\} = \sqrt{2}$.
$\inf\{x \in \mathbb{R} : x^2 \le 2\} = -\sqrt{2} \in \{x \in \mathbb{R} : x^2 \le 2\}$ and so
$\min\{x \in \mathbb{R} : x^2 \le 2\} = -\sqrt{2}$.

(8) $\sup\{0, 2, 10, 2010\} = 2010 \in \{0, 2, 10, 2010\}$ and so
$\max\{0, 2, 10, 2010\} = 2010$.
$\inf\{0, 2, 10, 2010\} = 0 \in \{0, 2, 10, 2010\}$ and so
$\min\{0, 2, 10, 2010\} = 0$.

(9) This set has elements $-\frac{2}{1}, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, \ldots$.
$\sup\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\} = \frac{3}{2} \in \{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{Z}\}$ and so
$\max\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\} = \frac{3}{2}$.
$\inf\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\} = -2 \in \{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{Z}\}$, and so
$\min\{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\} = -2$.

(10) $\sup\{x^2 : x \in \mathbb{R}\} = +\infty$, and $\max\{x^2 : x \in \mathbb{R}\}$ does not exist.

$\inf\{x^2 : x \in \mathbb{R}\} = 0 \in \{x^2 : x \in \mathbb{R}\}$ and so $\min\{x^2 : x \in \mathbb{R}\} = 0$.

(11) $\sup\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\} = 1 \notin \left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ and so

$\max\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ does not exist.

$\inf\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\} = 0 \in \left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ and so

$\min\left\{\frac{x^2}{1+x^2} : x \in \mathbb{R}\right\} = 0$.

Exercise 1.11.

(1) FALSE. (Take $S = \{1\}$. Then $u = 3$ is an upper bound of S, and although $u' = 2 < 3 = u$, $u' (= 2)$ is an upper bound of $\{1\} = S$.)

(2). TRUE. (If $\epsilon > 0$, then $u_* - \epsilon < u_*$, and so $u_* - \epsilon$ cannot be an upper bound of S).

(3) FALSE. ($\mathbb{N}$ has no maximum.)

(4) FALSE. ($\sup \mathbb{N} = +\infty \notin \mathbb{R}$.)

(5) TRUE. (Definition of maximum.)

Exercise 1.12

Since $\sup B$ is an upper bound of B, we have $b \leq \sup B$ for all $b \in B$, and also for all $a \in A$, since $A \subset B$. So $\sup B$ is an upper bound of A, and so by the definition of the least upper bound of A, we obtain $\sup A \leq \sup B$.

Exercise 1.13

Let $z \in A+B$. Then $z = x+y$ for some $x \in A$ and $y \in B$. Since $\sup A$, $\sup B$ are upper bounds for A, B, respectively, we have $x \leq \sup A$ and $y \leq \sup B$. Thus $z = x+y \leq \sup A + \sup B$. So $\sup A + \sup B$ is an upper bound for $A+B$. Hence $\sup(A+B) \leq \sup A + \sup B$.
(If either $\sup A$ or $\sup B$ is $+\infty$, then the inequality is trivial.)

Exercise 1.14

Let ℓ be a lower bound of S: $\forall x \in S$, $\ell \leq x$.

So $\forall x \in S$, $-x \leq -\ell$, in other words,

$\forall y \in -S$, $y \leq -\ell$.

Hence $-S$ is bounded above because $-\ell$ is an upper bound of $-S$.

Since S is nonempty, it follows that $\exists x \in S$, and so we obtain that $-x \in -S$. Hence $-S$ is nonempty.

As $-S$ is nonempty and bounded above, it follows that $\sup(-S)$ exists, by the least upper bound property of $\mathbb{R}$.

Since $\sup(-S)$ is an upper bound of $-S$, we have:

$$\forall y \in -S, \ y \leq \sup(-S)$$

that is, $\forall x \in S$, $-x \leq \sup(-S)$,

that is, $\forall x \in S$, $-\sup(-S) \leq x$.

So $-\sup(-S)$ is a lower bound of S.

Next we prove that $-\sup(-S)$ is the greatest lower bound of S. Suppose that ℓ' is a lower bound of S such that $-\sup(-S) < \ell'$. Then we have

$$\forall x \in S, \ -\sup(-S) < \ell' \leq x,$$

that is, $\forall x \in S$, $-x \leq -\ell' < \sup(-S)$,

that is, $\forall y \in -S$, $y \leq -\ell' < \sup(-S)$.

So $-\ell'$ is an upper bound of $-S$, and $-\ell' < \sup(-S)$, which contradicts the fact that $\sup(-S)$ is the least upper bound of $-S$. Hence $\ell' \leq -\sup(-S)$.

Consequently, $\inf S$ exists and $\inf S = -\sup(-S)$.

Exercise 1.16.

It is easy to show that $\{f(x) : x \in \mathbb{R}\}$ is bounded.

1° $\epsilon := \sup\limits_{x\in\mathbb{R}} f(x) > 0$.

Choose R such that if $|x| > R$, then $|f(x)| < \frac{\epsilon}{2}$.

By the extreme value theorem, $f|_{[-R,R]}$ assumes a maximum in $[-R, R]$, at say, $x_0 \in [-R, R]$.

But then $f(x_0) \geq \frac{\epsilon}{2}$ (otherwise $f(x) < \frac{\epsilon}{2}$ for all x, and so $\epsilon = \sup\limits_{x\in\mathbb{R}} f(x) \leq \frac{\epsilon}{2}$ a contradiction).

So $f(x_0) \geq f(x)$ $\forall\, x \in \mathbb{R}$

and so x_0 is a maximizer.

2° $\inf\limits_{x\in\mathbb{R}} f(x) < 0$.

Then $-\sup\limits_{x\in\mathbb{R}} (-f(x)) < 0$ i.e., $\sup\limits_{x\in\mathbb{R}} (-f(x)) > 0$, and by the result in 1°, $-f$ has a global maximizer.

Consequently f has a global minimizer.

3° $\neg\left[\sup\limits_{x\in\mathbb{R}} f(x) > 0\right]$ and $\neg\left[\inf\limits_{x\in\mathbb{R}} f(x) < 0\right]$.

Then

$$0 \leq \inf_{x\in\mathbb{R}} f(x) \leq f(x) \leq \sup_{x\in\mathbb{R}} f(x) \leq 0$$

and so $f(x) \equiv 0$.

So every point serves as a maximizer and a minimizer

Examples:

$f(x) = \dfrac{1}{1+x^2}$ $(x \in \mathbb{R})$ has a maximum at $x = 0$,

and $\inf\limits_{x\in\mathbb{R}} f(x) = 0$, but clearly $f(x) \neq 0$ $\forall\, x$. So f has no minimizer.

$-f$ has a minimizer, but no maximizer.

Exercise 1.17

The ball $B_i = \{ x \in \mathbb{R}^n : \|x\| < 1 \}$ is open.

(If $x \in B_i$, then $\|x\| < 1$. Define $r = 1 - \|x\|$. Then $B(x, r) \subset B_i$ since if $y \in B(x,r)$, then we have $\|x - y\| < r$ and so $\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < r + \|x\| = 1 - \|x\| + \|x\| = 1$,

Also the ball $B_e = \{ x \in \mathbb{R}^n : \|x\| > 1 \}$ is open.

(If $x \in B_e$, then $\|x\| > 1$. Define $r = \|x\| - 1$. Then $B(x, r) \subset B_e$ since if $y \in B(x, r)$, then we have $\|x - y\| < r$ and so $\|y\| = \|x - (x-y)\| \geq \|x\| - \|x - y\| > \|x\| - r = \|x\| - (\|x\| - 1) =$

Thus $B_e \cup B_i$ is open as well.

Hence $S^{n-1} = \mathbb{R}^n \setminus (B_e \cup B_i)$ is closed.

Clearly S^{n-1} is bounded. $(\forall x \in S^{n-1}, \ \|x\| \leq 1)$

Since S^{n-1} is closed and bounded, it is compact

Exercise 1.18.

The set of polynomials of degree 2010 is contained in the vector space of all polynomials of degree ≤ 2010, which we identify with a 2011 dimensional real vector space with the usual Euclidean norm. On the Euclidean unit sphere S_{2011}, consider the two continuous functions $f(p) = |p(0)|$ and $g(p) = \int_{-1}^{1} |p(x)|\, dx$. Since $g(p) \neq 0$ for all $p \in S_{2011}$, the ratio $\frac{f(p)}{g(p)}$ is a continuous function on the compact set S_{2011}, and hence achieves a maximum value C there. This is the required constant since $f(\alpha p) = \alpha f(p)$ and $g(\alpha p) = \alpha g(p)$ for all $\alpha > 0$ and all polynomials p.