Home assignment number 2, autumn 2017, in SF1811 Optimization.

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This assignment should be carried out in groups containing one or two persons. There should be one (short!) report per group, in which you describe "in your own words" how the problem was formulated and solved, and answer the given questions. Limited cooperation ("discussions") with other groups is allowed, but should in that case be mentioned in the beginning of the report. It is never allowed to copy parts of another groups report or matlab code! State your name, personal number and *email* address at the front of the report.
Your report should be uploaded to canvas not later than **17:00, December 15**. Both the report and your matlab code should be handed in as a separate files (e.g., a report as pdf file and code as m-files). It should be possible for David to, without any effort!, run your code and obtain the results you presented in the report. Some students may, partly by random, be selected for individual oral defense of their assignments. Those students will be contacted by email, so check your email regularly.
A correct report (and defense if asked for) gives you **2** bonus points on the exam.

In this home assignment you should formulate some portfolio optimization problems based on the Markowitz model as QP problems, solve them using Matlab, and present the results. The model can be described briefly as follows. Let
K = a given capital to invest in various assets (e.g. stocks) during the coming year,
n = number of different assets to invest in,
ξ_j = the rate of return of asset j. ξ_j is a *random variable*!
Note: If the market price of an asset changes from p_j (today) to q_j (one year from now) then the (annual) rate of return of the asset during the coming year is $(q_j - p_j)/p_j$.
Since q_j is not known (today), the rate of return is a random variable.
$\mu_j = \mathrm{E}\,[\xi_j]$ = the expected value of the rate of return of asset j,
$\sigma_j^2 = \mathrm{Var}(\xi_j)$ = the variance of the rate of return of asset j.
$\sigma_{ij} = \mathrm{Cov}(\xi_i, \xi_j)$ = the covariance of the rate of returns of assets i and j.

Let x_j = the fraction of K invested in asset j, where $0 \le x_j \le 1$ and $\sum_{j=1}^{n} x_j = 1$.

Then the rate of return of the portfolio is the random variable $\sum_{j=1}^{n} x_j \xi_j$.

The expected value $\mu(\mathbf{x})$ of this rate of return of the portfolio is

$$\mu(\mathbf{x}) = \mathrm{E}\,[\sum_{j=1}^{n} x_j \xi_j\,] = \sum_{j=1}^{n} x_j \mathrm{E}\,[\,\xi_j\,] = \sum_{j=1}^{n} x_j \mu_j = \boldsymbol{\mu}^{\mathsf{T}}\mathbf{x},$$

where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^\mathsf{T}$ and $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$, while the variance $\sigma^2(\mathbf{x})$ of the rate of return of the portfolio is

$$\sigma^2(\mathbf{x}) = \mathrm{E}\,[\,(\sum_{j=1}^{n} x_j \xi_j - \mu(\mathbf{x})\,)^2\,] = \mathrm{E}\,[\,(\sum_{j=1}^{n} x_j (\xi_j - \mu_j)\,)^2\,] =$$

$$\mathrm{E}\,[\,(\sum_{i=1}^{n} x_i(\xi_i - \mu_i))(\sum_{j=1}^{n} x_j(\xi_j - \mu_j))\,] = \mathrm{E}\,[\,\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j (\xi_i - \mu_i)(\xi_j - \mu_j)\,] =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \mathrm{E}\,[\,(\xi_i - \mu_i)(\xi_j - \mu_j)\,] = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij} = \mathbf{x}^\mathsf{T} \mathbf{C}\, \mathbf{x},$$

where $\mathbf{C}$ is the covariance matrix with elements σ_{ij}. Note that $\sigma_{ii} = \sigma_i^2$.
It is assumed that μ_j, σ_j^2 and σ_{ij} have been estimated for all i and j.
Since the variance $\sigma^2(\mathbf{x})$ is connected to "risk", a reasonable optimization problem based on the above assumptions is the following QP problem in the variable vector $\mathbf{x}$:

$$\begin{array}{rl} \text{minimize} & \mathbf{x}^\mathsf{T} \mathbf{C}\, \mathbf{x} \\ \text{subject to} & \boldsymbol{\mu}^\mathsf{T} \mathbf{x} = r, \\ & \mathbf{e}^\mathsf{T} \mathbf{x} = 1, \quad \text{where } \mathbf{e} = (1, \ldots, 1)^\mathsf{T}, \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \qquad (1)$$

for one or several different values of the right hand side r.

Use the following Matlab code for generating your vector $\boldsymbol{\mu}$ and matrix $\mathbf{C}$.

```
n=8;
dummystep=31*d1+d2;
rng('default');
for i=1:dummystep
dummy=rand;
end
Corr=zeros(n,n);
for i=1:n
for j=1:n
Corr(i,j)=(-1)^abs(i-j)/(abs(i-j)+1);
end
end
sigma=zeros(n,1);
mu=zeros(n,1);
sigma(1)=2;
mu(1)=3;
for i=1:n-1
sigma(i+1)=sigma(i)+2*rand;
mu(i+1)=mu(i)+1;
end
D=diag(sigma);
C2=D*Corr*D;
C=0.5*(C2+C2');
```

Here, `d1` is the day of the month the first member of your group was born, and `d2` is the corresponding number for the second member.
If there is no second member of the group then `d2` = `d1`.

Exercise 1.
Use the Matlab function `quadprog` to solve the above problem (1) for the following 25 different values of the right hand side r:

$r = 3.00,\ 3.25,\ 3.50,\ 3.75, \ldots\ldots,\ 8.50,\ 8.75,\ 9.00.$ Hint: Use a `for`-loop in Matlab.

Save the obtained 25 values of $\sigma(\mathbf{x}) = \sqrt{\mathbf{x}^\mathsf{T}\mathbf{C}\,\mathbf{x}}$ and $\mu(\mathbf{x}) = \boldsymbol{\mu}^\mathsf{T}\mathbf{x}$ in two vectors. Plot these vectors in a figure showing σ on the horizontal and μ on the vertical axis.

Exercise 2.
Repeat Exercise 1, including `quadprog` runs, but modified as follows:
Assume that it is not necessary to invest the whole capital K, and that the *not invested*
fraction $1 - \sum_j^n x_j$ of K can be saved without any return and without any "risk".
Motivate that this situation can be modelled by simply changing the constraint $\mathbf{e}^\mathsf{T}\mathbf{x} = 1$ in the problem (1) to $\mathbf{e}^\mathsf{T}\mathbf{x} \leq 1$. Then comment on the difference between your obtained new figure and your figure from Exercise 1.

Exercise 3.
Repeat Exercise 1, including `quadprog` runs, but now modified as follows:
Assume that the constraint $\boldsymbol{\mu}^\mathsf{T}\mathbf{x} = r$ in problem (1) is changed to $\boldsymbol{\mu}^\mathsf{T}\mathbf{x} \geq r$.
Give an interpretation of this change in the model, and motivate the difference between your obtained new figure and your figure from Exercise 1.

Exercise 4.
Repeat Exercise 1, including `quadprog` runs, but now modified as follows:
Assume that so called "short selling" is possible. This essentially means that it is possible to borrow assets today and immediatly sell them for today's market price. Then, at a future date agreed upon, (e.g. one year from today) the borrowed assets must be bought back, for the market price at that future date, and returned to the lender.
Motivate that this situation can be modelled by simply removing the constraints $\mathbf{x} \geq \mathbf{0}$ in the problem (1). Then comment briefly on the difference between your obtained new figure and your figure from Exercise 1.

Note: Short selling might be *dangerous* since there is in principles no upper bound on the possible loss (if the market price on the borrowed asset increases dramatically).

Comment on the plotting: It is recommended that you, instead of presenting four separate figures with one pair of vectors in each figure, present the following three figures:
Figure 1: The two pairs of vectors corresponding to Exercises 1 and 2.
Figure 2: The two pairs of vectors corresponding to Exercises 1 and 3.
Figure 3: The two pairs of vectors corresponding to Exercises 1 and 4.