Exercise 22.11

Let $X = \{ x : x_j \geq 0 \text{ for all } j = 1, \dots, n \}$.

Define $L : X \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ by

$$L(x, y) = \sum_{j=1}^{n} x_j^2 + y \left(b - \sum_{j=1}^{n} a_j x_j \right). \quad (x \in X, \ y \in \mathbb{R}).$$

The relaxed Lagrange problem (PR_y) is the following:

Given $y \geq 0$, minimize $x \mapsto L(x, y)$ on X, that is

$$(PR_y) : \begin{cases} \text{minimize} & \sum_{j=1}^{n} x_j^2 + y \left(b - \sum_{j=1}^{n} a_j x_j \right) \\ \text{subject to} & x_j \geq 0, \quad j = 1, \dots, n. \end{cases}$$

But

$$\begin{aligned} \sum_{j=1}^{n} x_j^2 + y \left(b - \sum_{j=1}^{n} a_j x_j \right) &= \sum_{j=1}^{n} \left(x_j^2 - y a_j x_j + \frac{1}{4} y^2 a_j^2 \right) \\ &\quad - \sum_{j=1}^{n} \frac{1}{4} y^2 a_j^2 + y b \\ &= \sum_{j=1}^{n} \left(x_j - \frac{1}{2} y a_j \right)^2 - \frac{1}{4} y^2 \sum_{j=1}^{n} a_j^2 + y b \end{aligned}$$

which is minimized when $x_j = \hat{x}_j(y) = \frac{1}{2} y a_j$, $j = 1, \dots, n$

and moreover since $y \geq 0$ and $a_j > 0$ for all j, it follows that $\hat{x}_j(y) \in X$.

The dual objective function is given by

$$\varphi(y) = L(\hat{x}(y), y) = -\frac{1}{4} y^2 \sum_{j=1}^{n} a_j^2 + y b$$

and this is maximized if

$$y = \hat{y} = \frac{2b}{\sum_{j=1}^{n} a_j^2} > 0.$$

Finally, set $\hat{x}_j = \hat{x}_j(\hat{y}) = \dfrac{b}{\sum_{j=1}^{n} a_j^2} a_j$, $j = 1, \dots, n$.

Then $\hat{x}_j \geq 0$ and $\sum_{j=1}^{n} a_j \hat{x}_j = b$. So $\hat{x}$ is feasible for the original problem, and the corresponding cost is $f(\hat{x}) = \varphi(\hat{y})$.

Hence $\hat{x}$ is a global optimal solution to the original problem.

Exercise 22.12

Let $X = \{x : x_j > 0 \text{ for all } j = 1, \dots, n\}$

Define $L : X \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ by

$$L(x, y) = -\sum_{j=1}^{n} \log x_j + y\left(\sum_{j=1}^{n} a_j x_j - b\right) \qquad (x \in X,\ y \in \mathbb{R})$$

The relaxed Lagrange problem (PR_y) is the following:

Given $y \geq 0$, minimize $x \longmapsto L(x, y)$ on X, that is

$$(PR_y): \begin{cases} \text{minimize} & -\sum_{j=1}^{n} \log x_j + y\left(\sum_{j=1}^{n} a_j x_j - b\right) \\ \text{subject to} & x_j > 0,\ j = 1, \dots, n \end{cases}$$

The function

$$x \overset{F}{\longmapsto} -\sum_{j=1}^{n} \log x_j + y\left(\sum_{j=1}^{n} a_j x_j - b\right)$$ can be decomposed into the sum of n 1-varible convex functions

$x_j \longmapsto -\log x_j + y a_j x_j - \frac{yb}{n}$ for $x_j > 0$.

These are minimized if $x_j = \frac{1}{y a_j} > 0$ for all $j = 1, \dots, n$.

So this x gives a local and hence a global optimal solution to (PR_y).

So we have $\hat{x}_j(y) = \frac{1}{y a_j}$, $j = 1, \dots, n$.

The dual objective function is given by

$$\varphi(y) = L(\hat{x}(y), y) = -\sum_{j=1}^{n} \log \frac{1}{y a_j} + y\left(\sum_{j=1}^{n} a_j \frac{1}{y a_j} - b\right)$$

$$= \sum_{j=1}^{n} \log a_j + n \log y + y\left(\frac{n}{y} - b\right)$$

$$= \sum_{j=1}^{n} \log a_j + n \log y + n - yb,$$

and this is maximized if

$$y = \hat{y} = \frac{n}{b} > 0.$$

Then $\hat{x}_j = \hat{x}_j(\hat{y}) = \dfrac{b}{n a_j}$, $j = 1, \dots, n$. Then $\hat{x}_j \geq 0$ for all j and

$$\sum_{j=1}^{n} a_j \hat{x}_j = n \cdot \frac{b}{n} = b$$

So this $\hat{x}$ is feasible for the original problem. Also, $\hat{y} \geq 0$ and $f(\hat{x}) = \varphi(\hat{y})$.

Hence $\hat{x}$ is a global optimal solution to the original problem.

Exercise 22.13

Let $X = \{x : x_j > 0$ for all $j = 1, \dots, n\}$.

Define $L: X \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ by

$$L(x,y) = \sum_{j=1}^{n} a_j x_j + y \left(\sum_{j=1}^{n} \frac{b_j}{x_j} - b_0 \right) \qquad (x \in X,\ y \in \mathbb{R}).$$

The relaxed Lagrange problem (PR_y) is the following:

Given $y \geq 0$, minimize $x \longmapsto L(x,y)$ on X, that is,

$$(PR_y): \begin{cases} \text{minimize } \sum_{j=1}^{n} a_j x_j + y \left(\sum_{j=1}^{n} \frac{b_j}{x_j} - b_0 \right) \\ \text{subject to } x_j > 0, \quad j = 1, \dots, n. \end{cases}$$

The function

$$x \overset{F}{\longmapsto} \sum_{j=1}^{n} a_j x_j + y \left(\sum_{j=1}^{n} \frac{b_j}{x_j} - b_0 \right) \text{ can be}$$

decomposed into the sum of the n convex 1-variable functions $x_j \longmapsto a_j x_j + y \frac{b_j}{x_j} - \frac{y b_0}{n}$,

which are minimized at $x_j = \sqrt{\frac{y b_j}{a_j}}$.

So $x_j = \hat{x}_j(y) = \sqrt{\frac{y b_j}{a_j}}$, for $j = 1, \dots, n$.

The dual objective function is

$$\varphi(y) = L(\hat{x}(y), y) = \sum_{j=1}^{n} \sqrt{a_j b_j}\,\sqrt{y} + y \left(\sum_{j=1}^{n} \frac{\sqrt{a_j b_j}}{\sqrt{y}} - b_0 \right)$$

This is maximized if

$$\frac{1}{\sqrt{y}} = \frac{b_0}{\sum_{j=1}^{n} \sqrt{a_j b_j}}.$$

Set $\hat{x}_j = \hat{x}_j(\hat{y}) = \sqrt{\frac{b_j}{a_j}} \Big/ \frac{b_0}{\sum_{i=1}^{n} \sqrt{a_i b_i}}$, $j = 1, \dots, n$

Then $\hat{x}_j > 0$ for all j and

$$\sum_{j=1}^{n} \frac{b_j}{x_j} = \left(\sum_{j=1}^{n} \frac{b_j}{\sqrt{b_j}} \sqrt{a_j} \right) \frac{b_0}{\sum_{i=1}^{n} \sqrt{a_i b_i}} = b_0.$$

So $\hat{x}$ is a feasible solution to the original problem.

Since $(\hat{x}, \hat{y})$ satisfy the global optimality conditions associated with (P), $\hat{x}$ is an optimal solution to (P

Exercise 22.14

Let $X = \mathbb{R}^n$. Define $L : X \times \mathbb{R}^1 \longrightarrow \mathbb{R}$ by

$$L(x, y) = \sum_{j=1}^n e^{c_j x_j} + y\left(b - \sum_{j=1}^n a_j x_j\right) \qquad (x \in X = \mathbb{R}^n,\ y \in \mathbb{R}).$$

The relaxed Lagrange problem (PR_y) is the following: Given $y \geq 0$, minimize $x \longmapsto L(x, y)$ on $X = \mathbb{R}^n$, that is,

$$(PR_y): \begin{cases} \text{minimize} & \sum_{j=1}^n e^{c_j x_j} + y\left(b - \sum_{j=1}^n a_j x_j\right) \\ \text{subject to} & x \in \mathbb{R}^n. \end{cases}$$

The function

$$x \overset{F}{\longmapsto} \sum_{j=1}^n e^{c_j x_j} + y\left(b - \sum_{j=1}^n a_j x_j\right)$$

is convex on $\mathbb{R}^n$, and the global minimum is obtained by solving $\nabla F(x) = 0$ i.e.,

$$[c_1 e^{c_1 x_1} - y a_1 \quad \cdots \quad c_n e^{c_n x_n} - a_n] = 0$$

i.e., $$x_j = \frac{1}{c_j} \log\left(\frac{y a_j}{c_j}\right) \in \mathbb{R}, \quad j = 1, \ldots, n$$

The dual objective function is

$$\varphi(y) = L(\hat{x}(y), y)$$

$$= \sum_{j=1}^n e^{\log\left(\frac{y a_j}{c_j}\right)} + y\left(b - \sum_{j=1}^n \frac{a_j}{c_j} \log\left(\frac{y a_j}{c_j}\right)\right)$$

$$= y\left(\sum_{j=1}^n \left(\frac{a_j}{c_j}\right)\right) + y\left(b - \sum_{j=1}^n \frac{a_j}{c_j} \log\left(\frac{y a_j}{c_j}\right)\right).$$

We have

$$\varphi'(y) = \sum_{j=1}^n \left(\frac{a_j}{c_j}\right) + b - \sum_{j=1}^n \frac{a_j}{c_j} \log \frac{a_j}{c_j} - \sum_{j=1}^n \frac{a_j}{c_j}(1 + \log y)$$

$$= b - \sum_{j=1}^n \frac{a_j}{c_j} \log \frac{a_j}{c_j} - \left(\sum_{j=1}^n \frac{a_j}{c_j}\right) \log y.$$

$= 0$

if $$\log y = \frac{b - \sum_{j=1}^n \frac{a_j}{c_j} \log \frac{a_j}{c_j}}{\sum_{j=1}^n \frac{a_j}{c_j}}.$$

So $y = \exp(\log y) > 0$.

Hence $$\hat{x}_j = \frac{1}{c_j}\left(\log \frac{a_j}{c_j} + \frac{b - \sum_{j=1}^n \frac{a_j}{c_j} \log \frac{a_j}{c_j}}{\sum_{j=1}^n \frac{a_j}{c_j}}\right), \quad j = 1, \ldots, n.$$

Exercise 22.15

(a) Let $X = \{ x : x_j \geq 0 \text{ for all } j = 1, \dots, n \}$.

Define $L : X \times \mathbb{R}^1 \to \mathbb{R}$ by

$$L(x, y) = \sum_{j=1}^{n} x_j^3 + y \left(b - \sum_{j=1}^{n} a_j x_j \right).$$

The relaxed Lagrange problem (PR_y) is the following:

Given $y \geq 0$, minimize $x \mapsto L(x, y)$ on X, that is,

$$(PR_y) : \begin{cases} \text{minimize} & \sum_{j=1}^{n} x_j^3 + y \left(b - \sum_{j=1}^{n} a_j x_j \right) \\ \text{subject to} & x_j \geq 0, \quad j = 1, \dots, n. \end{cases}$$

The function $x \xmapsto{F} \sum_{j=1}^{n} x_j^3 + y \left(b - \sum_{j=1}^{n} a_j x_j \right)$ which can be decomposed into the sum of n 1-variable convex functions $x_j \mapsto x_j^3 + \frac{yb}{n} - y a_j x_j$ on $x_j \geq 0$. These are minimized if $x_j = \hat{x}_j(y) = \sqrt{\frac{y a_j}{3}} \geq 0$ for all $j = 1, \dots, n$.

The dual objective function is given by

$$\varphi(y) = L(\hat{x}(y), y) = \left[\sum_{j=1}^{n} \left(\frac{a_j}{3} \right)^{3/2} \right] y^{3/2} + y \left[b - \left(\sum_{j=1}^{n} \frac{a_j^{3/2}}{\sqrt{3}} \right) y \right]$$

$$= \left(-\frac{2}{3} \cdot \frac{1}{\sqrt{3}} \sum_{j=1}^{n} a_j^{3/2} \right) y^{3/2} + yb,$$

and the dual optimization problem (D) is:

$$(D) : \begin{cases} \text{maximize} & \left(\frac{-2}{3\sqrt{3}} \sum_{j=1}^{n} a_j^{3/2} \right) y^{3/2} + yb, \\ \text{subject to} & y \geq 0. \end{cases}$$

(b) The solution $\hat{y}$ to (D) is given by

$$\frac{1}{\sqrt{3}} \sum_{j=1}^{n} a_j^{3/2} \cdot \sqrt{y} = b$$

i.e., $$\hat{y} = \frac{3b^2}{\left(\sum_{j=1}^{n} a_j^{3/2} \right)^2} > 0.$$

Then

$$\hat{x}_j := \hat{x}_j(\hat{y}) = \frac{\sqrt{a_j}\, b}{\left(\sum^{n} a_j^{3/2} \right)} > 0, \quad j = 1, \dots, n.$$

Exercise 22.16

Let

$$f(x) := x_1^4 + 2x_1x_2 + x_2^2 + x_3^8,$$
$$g_1(x) := (x_1-2)^2 + (x_2-2)^2 + (x_3-3)^2 - 6,$$
$$g_2(x) := x_1x_2x_3 - 10,$$
$$g_3(x) := 1 - x_1,$$
$$g_4(x) := -x_2,$$
$$g_5(x) := -x_3.$$

Then the problem (P) is

$$(P): \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \le 0, \quad i = 1,2,3,4,5, \\ & x \in \mathbb{R}^3. \end{cases}$$

We have with $\hat{x} = (1,1,1) \in \mathbb{R}^3$ that

$$\left.\begin{aligned} g_1(\hat{x}) &= 0, \\ g_2(\hat{x}) &= -9, \\ g_3(\hat{x}) &= 0, \\ g_4(\hat{x}) &= 1, \text{ and} \\ g_5(\hat{x}) &= -1. \end{aligned}\right\} \quad (*)$$

So $\hat{x}$ is feasible for (P).

Now we want to find a $\hat{y}$ such that $\hat{x}$ is a minimizer of

$$x \longmapsto L(x, \hat{y}),$$

where

$$L(x,\hat{y}) := f(x) + \sum_{i=1}^{5} \hat{y}_i g_i(x), \qquad x \in \mathbb{R}^3.$$

Moreover, $\hat{y} \geq 0$ and $\hat{y}^T g(\hat{x}) = 0$. But this implies (in light of $(*)$) that $y_2 = y_4 = y_5 = 0$.

So

$$L(x,\hat{y}) = f(x) + \hat{y}_1 g_1(x) + \hat{y}_3 g_3(x).$$

But since $\hat{x}$ must be a global minimizer, we must have $\nabla L(x,\hat{y})\big|_{x=\hat{x}} = 0$.

This gives

$$\nabla f(\hat{x}) + \hat{y}_1 \nabla g_1(\hat{x}) + \hat{y}_3 \nabla g_3(\hat{x}) = 0.$$

We have

$$\nabla f(x) = [4x_1^3 + 2x_2 \quad 2x_1+2x_2 \quad 8x_3^7]$$
$$\nabla g_1(x) = [2(x_1-2) \quad 2(x_2-2) \quad 2(x_3-3)]$$
$$\nabla g_3(x) = [-1 \quad 0 \quad 0]$$

Hence

$$[6 \quad 4 \quad 8] + \hat{y}_1[-2 \quad -2 \quad -4] + \hat{y}_2[-1 \quad 0 \quad 0] = 0 \qquad (**)$$

i.e.,

$$2\hat{y}_1 + \hat{y}_2 = 6$$
$$2\hat{y}_1 = 4$$
$$4\hat{y}_1 = 8$$

which has the solution $\hat{y}_1 = 2$, $\hat{y}_2 = 2$.

We have

$$F(x) := L(x,\hat{y}) = f(x) + 2g_1(x) + 2g_3(x)$$
$$= x_1^4 + 2x_1x_2 + x_2^2 + x_3^8 + 2(x_1-2)^2 + 2(x_2-2)^2 + 2(x_3-2)^2 - 12 + 2 - 2x_1$$
$$= x_1^4 + (x_1+x_2)^2 + x_3^8 + 2(x_2-2)^2 + 2(x_3-2)^2 - x_1^2 + 2x_1^2 - 8x_1 + 8 - 12 + 2 - 2x_1$$
$$= x_1^4 + (x_1+x_2)^2 + x_3^8 + 2(x_2-2)^2 + 2(x_3-2)^2 + x_1^2 - 10x_1 + 25 + (8-12+2-25).$$
$$= x_1^4 + (x_1+x_2)^2 + x_3^8 + 2(x_2-2)^2 + 2(x_3-2)^2 + (x_1-5)^2 + (8-12+2-25).$$

Since F is a sum of convex functions, F is convex

Also we know (from $(**)$) that

$$\nabla F(\hat{x}) = 0.$$

So $\hat{x}$ is a global minimizer of F.

So we have found $(\hat{x},\hat{y}) \in X \times \mathbb{R}^5$ (where $X := \mathbb{R}^3$)

such that

(1) $L(\hat{x},\hat{y}) = \min_{x\in X} L(x,\hat{y}) \left(= \min_{x\in\mathbb{R}^3} F(x)\right)$,

(2) $g(\hat{x}) \leq 0$

(3) $\hat{y} \geq 0$

(4) $\hat{y}^T g(\hat{x}) = 0$.

Hence $\hat{x}$ is an optimal solution for (P).

Exercise 22.17.

Lemma 22.6'. If x is a feasible solution to (P) and $y \geq 0$, then $\varphi(y) \leq f(x)$.

Proof. Let x be a feasible solution to (P), that is, $g(x) \leq 0$ and $x \in X$. Then we have

$$\varphi(y) \leq f(x) + \underbrace{y^T}_{\geq 0} \underbrace{g(x)}_{\leq 0} \leq f(x). \qquad \square$$

Lemma 22.7' If (1) $\hat{x}$ is a feasible solution to (P),
(2) $\hat{y} \geq 0$ and
(3) $\varphi(\hat{y}) = f(\hat{x})$,

then $\hat{x}$ is an optimal solution to (P) and $\hat{y}$ is an optimal solution to (D).

Proof. Let x be a feasible solution to (P) and $y \geq 0$. From Lemma 22.6', we have $\varphi(y) \leq f(\hat{x})$ and $\varphi(\hat{y}) \leq f(x)$. If these are combined with $\varphi(\hat{y}) = f(\hat{x})$, we obtain that

$$\varphi(y) \leq f(\hat{x}) = \varphi(\hat{y}) \quad \text{and}$$
$$f(\hat{x}) = \varphi(\hat{y}) \leq f(x).$$

But this means that $\hat{y}$ is an optimal solution to (D) and $\hat{x}$ is an optimal solution to (P). $\square$

Theorem 22.8' $(\hat{x}, \hat{y}) \in X \times \mathbb{R}^m$ satisfy the global optimality conditions associated with (P) iff

(1) $\hat{x}$ is an optimal solution to (P),

(2) $\hat{y}$ is an optimal solution to (D), and

(3) $\varphi(\hat{y}) = f(\hat{x})$.

Proof. Suppose first that $(\hat{x}, \hat{y}) \in X \times \mathbb{R}^m$ satisfy the global optimality conditions associated with (P). Then from Theorem 22.4, it follows that $\hat{x}$ is optimal for (P). Also

$$\varphi(\hat{y}) = \inf_{x \in X} L(x, \hat{y}) = L(\hat{x}, \hat{y}) = f(\hat{x}) + \hat{y}^T g(\hat{x}) = f(\hat{x}) + 0 = f(\hat{x}),$$

which proves (3). Also by Lemma 22.7', $\hat{y}$ is an optimal solution to (D).

Now suppose that $\hat{x}$ is an optimal solution to (P), $\hat{y}$ is an optimal solution to (D), and $\varphi(\hat{y}) = f(\hat{x})$.
Now $\hat{x}$ being a feasible solution to (P), satisfies (2).
Also, since $\hat{y}$ is feasible for (D), $\hat{y} \geq 0$ and so (3) is also satisfied. We have

$$\varphi(\hat{y}) = \inf_{x \in X} \left(f(x) + \hat{y}^T g(x)\right) \leq f(\hat{x}) + \underbrace{\hat{y}^T}_{\geq 0} \underbrace{g(\hat{x})}_{\leq 0} \leq f(\hat{x}).$$

But $\varphi(\hat{y}) = f(\hat{x})$ and so

$$\varphi(\hat{y}) = f(\hat{x}) + \hat{y}^T g(\hat{x}) = f(\hat{x})$$

So that $\hat{y}^T g(\hat{x}) = 0$ i.e., (4) is satisfied.
Also

$$\varphi(\hat{y}) = \inf_{x \in X} \left(f(x) + \hat{y}^T g(x)\right) = f(\hat{x}) + \hat{y}^T g(\hat{x}) = L(\hat{x}, \hat{y})$$

and so $\varphi(\hat{y}) = \min_{x \in X} \left(f(x) + \hat{y}^T g(x)\right) = \min_{x \in X} L(x, \hat{y}) = L(\hat{x}, \hat{y})$
So (1) is also satisfied. $\square$

If $f(x) = c^T x$, $g(x) = b - Ax$ and $X = \{x \in \mathbb{R}^n : x \geq 0\}$
then $L(x,y) = c^T x + y^T (b - Ax) = (c^T - y^T A) x + y^T b$.

$\underline{1^\circ}$ $c^T - y^T A < 0$ i.e., $c^T - y^T A \leq 0$ and $\exists i$ s.t. $(c^T - y^T A)_i < 0$.
Take $x(t) = t(c^T - y^T A)_i e_i$, $t > 0$.
Then $x(t) \in X$.
Moreover $L(x(t), y) = -t\left((c^T - y^T A)_i\right)^2 + y^T b \longrightarrow -\infty$ as $t \to \infty$
So $\inf_{x \in X} L(x,y) = -\infty$, i.e., $\varphi(y) = -\infty$.

$\underline{2^\circ}$ $c^T - y^T A \geq 0$
Then $L(x,y) = \underbrace{(c^T - y^T A)}_{\geq 0} \underbrace{x}_{\geq 0} + y^T b \geq y^T b$.

So $\inf_{x \in X} L(x,y) = y^T b = \varphi(y)$.

So the dual problem is

$$\begin{cases} \text{maximize} & \varphi(y) \\ \text{subject to} & y \geq 0 \end{cases} \quad \text{i.e.,} \quad \begin{cases} \text{maximize} & y^T b \\ \text{subject to} & y \geq 0, \\ & A^T y \leq c \end{cases}$$

Exercise 22.18

Let $L(x, y) = f(x) + y^T g(x)$

$$= -6x_1 - 4x_2 - 2x_3 + y_1(x_1^2 + x_2^2 - 2) + y_2(x_1^2 + x_3^2 - 2) + y_3(x_2^2 + x_3^2 - 2)$$

Then $\varphi(y) = \min_{x \in \mathbb{R}^3} L(x, y)$. If $\hat{y} = (1,1,1)$, then

$$L(x, \hat{y}) = -6x_1 - 4x_2 - 2x_3 + x_1^2 + x_2^2 - 2 + x_1^2 + x_3^2 - 2 + x_2^2 + x_3^2 - 2$$

$$= -6x_1 - 4x_2 - 2x_3 + 2x_1^2 + 2x_2^2 + 2x_3^2 - 6.$$

$$= \frac{1}{2} x^T H x + c^T x + c_0$$

where $H = 4I$, $c = [-6 \ \ -4 \ \ -2]^T$ and $c_0 = -6$.

Thus $x \mapsto L(x, \hat{y})$ has a minimizer $\hat{x}$ iff

$$H\hat{x} = -c$$

i.e., $4I\hat{x} = -\begin{bmatrix} -6 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$ and so $\hat{x} = \begin{bmatrix} 3/2 \\ 1 \\ 1/2 \end{bmatrix}$.

But then

$$\varphi(\hat{y}) = \frac{1}{2}\hat{x}^T 4I \hat{x} + c^T\hat{x} + c_0$$

$$= 2 \cdot \hat{x}^T\hat{x} + -6 \cdot \frac{3}{2} - 4 \cdot 1 + -2 \cdot \frac{1}{2} - 6$$

$$= 2\left(\frac{9}{4} + 1 + \frac{1}{4}\right) - 9 - 4 - 1 - 6$$

$$= 5 + 2 - 20 = -13$$

The result in Exercise 21.18

makes us guess that $\left.\left(\frac{2-c_1}{4}, \frac{-2-c_1}{4}, \frac{6+c_1}{4}\right)\right|_{c_1=-6}$ is

optimal for the dual problem, i.e., $\tilde{y} = (2, 1, 0)$.

We have

$$L(x, \tilde{y}) = -6x_1 - 4x_2 - 2x_3 + 2x_1^2 + 2x_2^2 - 4 + x_1^2 + x_3^2 - 2$$

$$= 3x_1^2 + 2x_2^2 + x_3^2 - 6x_1 - 4x_2 - 2x_3 - 6,$$

and the map $x \mapsto L(x, \tilde{y})$ has a minimizer $\tilde{x}$ iff

$$\tilde{x} = \begin{bmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Also $\varphi(\tilde{y}) = L(\tilde{x}, \tilde{y}) = 3 + 2 + 1 - 6 - 4 - 2 - 6 = -12 > \varphi(\hat{y}) =$

$\hat{y}$ cannot be an optimal soln. to (

Exercise 22.19

Define $L(x,y) = f(x) + y_1 g_1(x) + y_2 g_2(x)$

$$= x_1^2 - x_1x_2 + x_2^2 + x_3^2 - 2x_1 + 4x_2$$
$$+ y_1(-x_1 - x_2) + y_2(1 - x_3).$$
$$= x_1^2 + x_2^2 + x_3^2 - x_1x_2 + (-y_1 - 2)x_1 + (4 - y_1)x_2 - y_2x_3 + y$$

Then $L(x,y) = \frac{1}{2} x^T H x + c^T x + c_0$,

where

$$H := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad c := \begin{bmatrix} -y_1 - 2 \\ 4 - y_1 \\ -y_2 \end{bmatrix}, \quad c_0 = +y_2.$$

The map $x \longmapsto L(x,y) : \mathbb{R}^3 \to \mathbb{R}$ has a minimum at $\hat{x}$

iff $H\hat{x} = -c$

i.e., $$\hat{x} = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 2/3 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} y_1 + 2 \\ y_1 - 4 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_1 - 2 \\ y_2/2 \end{bmatrix}$$

So $\varphi(y) = L(\hat{x}(y), y)$

$$= \frac{1}{2} \hat{x}^T \underbrace{H\hat{x}}_{=-c} + c^T \hat{x} + c_0$$

$$= +\frac{1}{2} c^T \hat{x} + c_0$$

$$= -\frac{1}{2}(y_1 + 2)y_1 + \frac{1}{2}(4 - y_1)(y_1 - 2) - \frac{1}{2} y_2 \frac{y_2}{2} + y_2$$

$$= -y_1^2 + 2y_1 - \frac{y_2^2}{4} + y_2 - 4$$

Thus the dual problem is:

$$(D_c) : \begin{cases} \text{maximize} & -y_1^2 + 2y_1 - \frac{y_2^2}{4} + y_2 - 4, \\ \text{subject to} & y_1 \geq 0, \\ & y_2 \geq 0. \end{cases}$$

$-1 + \cancel{2} - 1 + \cancel{2} -$

Taking $\hat{x} = (1, -1, 1)$ and $\hat{y} = (1, 2)$, we see that

(1) $\hat{x}$ is feasible for (P_c)

(2) $\hat{y} \geq 0$

(3) $f(\hat{x}) = -2 = \varphi(\hat{y})$.

So $\hat{y}$ is an optimal solution to (D_c) and $\hat{x}$ is an optimal solution to (P_c)