EXAM FOR OPTIMIZATION SF1811/SF1831/SF1841

MARCH 19, 2011

Examiner: Amol Sasane (Phone: 790 7320)
Writing time allowed: 0800-1300
Writing material: Pen or pencil, eraser and a ruler is allowed. Calculators are **not** allowed! A formula sheet is provided.
Instructions: Motivate your answers carefully. Write your name on each page of the solutions you hand in and number the pages. Write the solutions to the different questions 1,2,3,4,5 on separate sheets. A passing grade is guaranteed for 25 points (including bonus points from the voluntary home assignments).

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(1) (a) Consider the following linear programming problem (NFP):

$$\text{(NFP)} : \begin{cases} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 5 \\ -4 \\ -3 \\ -3 \end{bmatrix}, \quad c = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 2 \\ 2 \\ 5 \\ 7 \end{bmatrix}.$$

From the special structure of the matrix A and the vector b, it follows that the problem (NFP) is a balanced network flow problem.

(i) Draw the corresponding network.

(ii) Verify that

$$\widehat{x} = \begin{bmatrix} 2 & 3 & 7 & 0 & 3 & 0 & 0 \end{bmatrix}^\top$$

is an optimal solution to the problem (NFP) by using the method for solving balanced network flow problems.

(6 points)

(b) Consider the matrix

$$H = \begin{bmatrix} 1 & 10 & 100 \\ 10 & 100 & 1000 \\ 100 & 1000 & 10000 \end{bmatrix}.$$

(i) Using $LDL^\top$-factorization, determine if H is positive semidefinite or not.

(ii) Verify that

$$c = \begin{bmatrix} 10 & -1 & 0 \end{bmatrix}^\top$$

belongs to the kernel of H.

(iii) Consider the unconstrained quadratic optimization problem

$$(\mathrm{Q}) : \left\{ \begin{array}{ll} \text{minimize} & \frac{1}{2} x^\top H x + c^\top x \\ \text{subject to} & x \in \mathbb{R}^3. \end{array} \right.$$

Does (Q) have an optimal solution? Justify your answer.

(4 points)

(2) (a) Consider the following linear programming problem:

$$(\mathrm{LP}) : \left\{ \begin{array}{ll} \text{minimize} & -x_1 - 2x_2 + 2x_3 \\ \text{subject to} & x_1 \phantom{{}+x_2} + 3x_3 + x_4 \phantom{{}+x_5} = 7, \\ & x_1 + x_2 - x_3 \phantom{{}+x_4} + x_5 = 7, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_3 \geq 0, \\ & x_4 \geq 0, \\ & x_5 \geq 0. \end{array} \right.$$

(i) Using the simplex method, find an optimal solution to (LP). Start with x_4 and x_5 as the basic variables.

(ii) Write down the dual problem to (LP).
Using the calculation done in part (i) above, write down an optimal solution to the dual problem.

(6 points)

(b) Determine if the following statements are true or false. All the statements below refer to linear programming problems in the standard form:

$$\left\{ \begin{array}{ll} \text{minimize} & c^\top x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \right.$$

where the variable x takes values in $\mathbb{R}^n$, the vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and the matrix $A \in \mathbb{R}^{m \times n}$ has rank $m < n$.

(i) For every problem of the form above, all basic feasible solutions have always exactly $n - m$ components equal to zero.

(ii) For each problem of the form above, if for a basic feasible solution $\overline{x}$ the reduced costs vector r_ν of the nonbasic variables is not ≥ 0, then $\overline{x}$ cannot be optimal.

(iii) If by using the simplex method for solving a linear programming problem of the type above, we have found an optimal solution $\widehat{x}$, then $\widehat{x}$ is always a basic feasible solution.

(iv) For each problem of the type above, the number of basic feasible solutions is always exactly equal to the number of basic tuples.

(4 points)

(3) (a) Find an optimal solution to the quadratic optimization problem

$$\begin{cases} \text{minimize} & 5x_1^2 + 4x_1x_2 + x_2^2 \\ \text{subject to} & 3x_1 + 2x_2 = 5. \end{cases}$$

using the nullspace method. Justify your answer.

(3 points)

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \log(1 + e^{-2x}) + x \quad \text{for } x \in \mathbb{R}.$$

(i) Show that $f''(x) > 0$ for all $x \in \mathbb{R}$.

(ii) If $x^{(0)}, x^{(1)}, x^{(2)}, \dots$ is the sequence of points generated by Newton's method for finding a global minimizer for f, then show that

$$x^{(k+1)} = x^{(k)} - \frac{1}{2}\sinh(2x^{(k)}), \quad (k = 0, 1, 2, \dots)$$

where for a real θ, $\sinh\theta := (e^\theta - e^{-\theta})/2$.

(iii) Show that if the initial point is such that the sequence $x^{(0)}, x^{(1)}, x^{(2)}, \dots$ is convergent, then the limit of the sequence must be 0.

(4 points)

(c) (i) What does it mean to say that a subset C of $\mathbb{R}^n$ is a convex set?

(ii) Is $\{x \in \mathbb{R}^3 : x_1^8 + x_2^{12} + x_3^{16} \leq 2011\}$ a convex subset of $\mathbb{R}^3$?

(3 points)

(4) (a) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2x_3, \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

(i) Show that the set of values of f on $\mathbb{R}^3$ is not bounded below.
Hint: Consider a curve $t \mapsto p(t) \in \mathbb{R}^3$ such that $f(p(t)) \to -\infty$ as $t \to \infty$.

(ii) Verify that the gradient of f at the point $v := (1, 1, 1)$ is zero.

(iii) Is v a local minimizer for f? Justify your answer.

(5 points)

(b) Solve the optimization problem

$$\begin{cases} \text{minimize} & (x_1 - 2)^2 + x_2^2 \\ \text{subject to} & x_1^2 + x_2^2 = 1 \end{cases}$$

using the method of Lagrange multipliers.

(5 points)

(5) (a) Consider the following problem:

$$\begin{cases} \text{minimize} & x_1 x_2 \\ \text{subject to} & x_1^2 + x_2^2 \le 1 \\ & x_1 + x_2 \ge 1. \end{cases}$$

(i) Sketch the feasible set.
(ii) Is the problem a convex optimization problem?
(iii) Sketch the following level sets

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = V\}$$

for $V = \frac{1}{2}$, $V = \frac{1}{4}$, and $V = 0$ in the same figure.
(iv) Write down the KKT-conditions associated with this optimization problem, and verify that the point $\widehat{x} := (1, 0)$ satisfies them.
Is $\widehat{x}$ an optimal solution to the problem? Justify your answer.

(6 points)

(b) Using the method of Lagrange relaxation, find the dual (D) to the problem

$$(\mathrm{P}) : \begin{cases} \text{minimize} & x^2 \\ \text{subject to} & x \ge 1. \end{cases}$$

Using duality theory, find an optimal solution to the dual problem (D), and an optimal solution to the primal problem (P). Justify your answers.

(4 points)

End of the exam.