KTH Matematik

Exam in SF1811 Optimization.
April 10, 2017, 14:00–19:00.

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2016. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different exercises 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

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1. In (a) and (b) below you should use the *simplex method* to solve two LP problems, while (c) deals with their corresponding dual problems. Both problems are on the standard form

$$\begin{aligned} &\text{minimize} && \mathbf{c}^{\mathsf{T}}\mathbf{x} \\ &\text{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & && \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

with a non-empty feasible region $\mathcal{F} = \{\mathbf{x} \in I\!R^4 \mid \mathbf{A}\mathbf{x} = \mathbf{b} \text{ and } \mathbf{x} \geq \mathbf{0}\}$.
In each of (a) and (b) below, your result should either be an optimal solution $\hat{\mathbf{x}}$ to the problem, or (if no optimal solution exists) a half-line on the parameter form $\mathbf{x}(t) = \bar{\mathbf{x}} + t\,\mathbf{d}$, where $\bar{\mathbf{x}}$ and $\mathbf{d}$ are fixed vectors in $I\!R^4$, and t is a non-negative parameter. This half-line should satisfy $\mathbf{A}\mathbf{x}(t) = \mathbf{b}$ and $\mathbf{x}(t) \geq \mathbf{0}$ for all $t \geq 0$, and $\mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \to -\infty$ when $t \to +\infty$.

(a) In the first problem, the data are as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{c}^{\mathsf{T}} = (\,2,\ 2,\ 3,\ -4\,).$$

Start the simplex method with x_1 and x_2 as basic variables. .... (5p)

(b) In the second problem, the data are as follows:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & 1 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{c}^{\mathsf{T}} = (\,2,\ 2,\ 3,\ -6\,).$$

Again, start the simplex method with x_1 and x_2 as basic variables. ...... (5p)

(c) For each of the above two problems, formulate the corresponding dual problem, and illustrate the constraints of this dual problem in a large, carefully drawn figure with the first dual variable y_1 on the horizontal axis and the second dual variable y_2 on the vertical. Also indicate the feasible region (if it exists) in the figure. . . . . (3p)

Note that the only difference between the two problems is that the last component in the cost vector $\mathbf{c}$ is -4 in the first problem and -6 in the second problem.

2. This exercise deals with a minimum cost flow problem, MCFP, in a certain network with four nodes and five arcs. There are two source nodes, node 1 and node 2, with given supplies s_1 units (node 1) and s_2 units (node 2), and two sink nodes, node 3 and node 4, with given demands d_3 units (node 3) and d_4 units (node 4).
It is assumed that $s_1 > 0,\ s_2 > 0,\ d_3 > 0,\ d_4 > 0$ and $s_1 + s_2 = d_3 + d_4 = 10$.
This can equivalently be expressed as follows, where u and v are given numbers:

$$s_1 = u,\ \ s_2 = 10-u,\ \ d_3 = v,\ \ d_4 = 10-v,\ \ 0<u<10 \ \text{ and } \ 0<v<10. \qquad (1)$$

All arcs are directed, and the set of arcs is given by $\{(1,2),(1,3),(2,3),(2,4),(3,4)\}$, where (i,j) denotes an arc *from* node i *to* node j.

The corresponding MCFP can be formulated as an LP problem on the form:
minimize $\mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, where the variable vector is
$\mathbf{x} = (x_{12}, x_{13}, x_{23}, x_{24}, x_{34})^\mathsf{T}$, with x_{ij} = the flow from node i to node j in (i,j).
The cost c_{ij} per unit flow in (i,j) is equal to 1 in all the arcs.
Thus, the cost vector is given by $\mathbf{c} = (c_{12}, c_{13}, c_{23}, c_{24}, c_{34})^\mathsf{T} = (1,1,1,1,1)^\mathsf{T}$.

Since the total supply equals the total demand, it is a balanced network flow problem, and then each *spanning tree* in the network corresponds to a basic solution to the MCFP problem.

(a) Illustrate the network in a figure, and write down, in details, the corresponding matrix $\mathbf{A}$ and right hand side vector $\mathbf{b}$ (expressed in u and v from (1)). . (3p)

(b) Consider the spanning tree defined by the arc set $T_1 = \{(1,2),(2,3),(3,4)\}$.
Calculate the corresponding basic solution $\mathbf{x}$, expressed in u and v from (1), and show that $\mathbf{x}$ is a feasible solution to the MCFP. Then show that $\mathbf{x}$ is *not* an optimal solution, by calculating a better solution $\tilde{\mathbf{x}}$, expressed in u and v.
($\tilde{\mathbf{x}}$ need not be optimal, but it should be a feasible solution to the MCFP and have a lower objective value than $\mathbf{x}$.) . . . . (3p)

(c) Consider the spanning tree defined by the arc set $T_2 = \{(1,3),(2,3),(2,4)\}$.
Calculate the corresponding basic solution $\hat{\mathbf{x}}$, expressed in u and v from (1).
Assume first that the given numbers u and v in (1) satisfies $0<u<v<10$.
Then show that $\hat{\mathbf{x}}$ is a unique optimal solution to the considered MCFP.
Assume next that the given numbers u and v in (1) satisfies $0<v<u<10$.
Then show that $\hat{\mathbf{x}}$ is *not* an optimal solution to the MCFP. . . . . (3p)

3. Consider the following quadratic optimization problem P in the variable vector $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T} \in I\!R^3$:

$$\begin{array}{lll} \text{P:} & \text{minimize} & f(\mathbf{x}) = \frac{1}{2}\,(\,x_1^2 + x_2^2 + x_3^2 - 2\,Cx_2x_3\,) \\ & \text{subject to} & x_1 + 2x_2 - 3x_3 = 10, \\ & & 3x_1 + \;x_2 - 4x_3 = 10, \end{array}$$

where $C \in I\!R$ is a given constant (which may be positive, negative or zero).

(a) Assume first that $C = 0$.
Then use a nullspace method to calculate an optimal solution $\hat{\mathbf{x}}$ to P.
Also calculate numbers (Lagrange multipliers) u_1 and u_2 which together with your calculated $\hat{\mathbf{x}}$ satisfy the Lagrange optimality conditions for P. . (5p)

(b) For which values on the constant C is there a *unique* optimal solution to P? For these values on C, calculate the optimal solution $\hat{\mathbf{x}}$ (which may or may not depend on C) and the optimal value $f(\hat{\mathbf{x}})$ of P. .... (3p)

(c) Is there any value on C for which there is an infinite number of optimal solutions to P? In that case, for this value on C, give an explicit expression for the set of optimal solutions to P, and the optimal value of P. .... (2p)

4. In this exercise, the following four functions h_i are given, where $\mathbf{x} = (x_1, x_2)^\mathsf{T} \in I\!R^2$.

$$\begin{aligned} h_1(\mathbf{x}) &= (x_1 + 2)^2 + (x_2 + 1)^2 - 4, \\ h_2(\mathbf{x}) &= (x_1 - 2)^2 + (x_2 + 1)^2 - 4, \\ h_3(\mathbf{x}) &= (x_1 + 1)^2 + (x_2 - 2)^2 - 4, \\ h_4(\mathbf{x}) &= (x_1 - 1)^2 + (x_2 - 2)^2 - 4, \end{aligned}$$

(a) Calculate an optimal solution $\tilde{\mathbf{x}} \in I\!R^2$ to the following problem:
minimize $h_1(\mathbf{x})+h_2(\mathbf{x})+h_3(\mathbf{x})+h_4(\mathbf{x})$ subject to $\mathbf{x} \in I\!R^2$.
Use your result to show that $h_1(\mathbf{x})+h_2(\mathbf{x})+h_3(\mathbf{x})+h_4(\mathbf{x}) > 0$ for all $\mathbf{x} \in I\!R^2$, and that there is no solution to the system $h_i(\mathbf{x}) = 0,\; i = 1, 2, 3, 4.$ ..... (3p)

(b) Then one would like to solve the following non-linear least squares problem:

$$\text{minimize} \;\; f(\mathbf{x}) = \tfrac{1}{2}\,(\,h_1(\mathbf{x})^2 + h_2(\mathbf{x})^2 + h_3(\mathbf{x})^2 + h_4(\mathbf{x})^2).$$

Consider the vector $\mathbf{x} = \mathbf{0}$, with $f(\mathbf{0}) = \frac{1}{2}\,(1 + 1 + 1 + 1) = 2$. Is this
(i) a global optimal solution?
(ii) a local but not global optimal solution?
(iii) not even a local optimal solution?
Motivate your answer carefully. .... (4p)

(c) Is the above function f a convex function on $I\!R^2$? Motivate carefully. ... (3p)

Hint: In (c), you may use that $h_1(\mathbf{x})+h_2(\mathbf{x})+h_3(\mathbf{x})+h_4(\mathbf{x}) > 0$ for all $\mathbf{x} \in I\!R^2$, even if you failed to solve (a).

5. Let S be the following set in $I\!R^3$:

$$S = \{\mathbf{x} = (x_1, x_2, x_3)^{\mathsf{T}} \in I\!R^3 \mid |x_1|+|x_2|+|x_3| \leq 1\}.$$

S is in fact the set of solutions to 8 linear inequality constraints:
$x_1+x_2+x_3 \leq 1,\ x_1+x_2-x_3 \leq 1,\ x_1-x_2+x_3 \leq 1,\ \dots\ , -x_1-x_2-x_3 \leq 1.$

(a) Let $\mathbf{q} = (-0.5,\ 0.4, -0.4)^{\mathsf{T}}$ and consider the following problem P:

$$\text{P:} \quad \text{minimize} \quad \tfrac{1}{2}\,\|\,\mathbf{x}-\mathbf{q}\,\|^2 \qquad \text{subject to} \quad \mathbf{x} \in S,$$

where, as usual, $\|\,\mathbf{x}-\mathbf{q}\,\|^2 = (\mathbf{x}-\mathbf{q})^{\mathsf{T}}(\mathbf{x}-\mathbf{q})$.
Use Lagrange relaxation (with respect to the linear inequality constraints) and the *global optimality conditions* to show that $\hat{\mathbf{x}} = (-0.4,\ 0.3, -0.3)^{\mathsf{T}}$ is an optimal solution to P. .... (4p)

(b) Assume now instead that $\mathbf{q} = (-0.8,\ 0.6, -0.1)^{\mathsf{T}}$.
Show, by using the same technique as above, that $\hat{\mathbf{x}} = (-0.6,\ 0.4,\ 0)^{\mathsf{T}}$ is an optimal solution to P. .... (4p)

Good luck!