KTH Matematik

Exam in SF1811/SF1841 Optimization.
Monday, August 18, 2014, 8:00–13:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2013. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
(This is important since the exams are split up during the corrections.)

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1. (a) Consider the following LP problem which we call P:

$$\begin{aligned} \text{P}: \quad & \text{minimize} \quad \mathbf{c}^\mathsf{T}\mathbf{x} \\ & \text{subject to} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \\ & \qquad\qquad\quad \mathbf{x} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} 25 \\ -15 \\ -10 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

It follows from the structure of the matrix $\mathbf{A}$ that P is in fact a minimum cost flow problem in a certain network. Illustrate that network in a figure.
Then verify that $\hat{\mathbf{x}} = (15,\, 0,\, 10,\, 0,\, 0,\, 0)^\mathsf{T}$ is an optimal solution to P.
Is there any other optimal solution to P? In that case, calculate such an optimal solution. .... (5p)

(b) Consider the given matrix $\mathbf{B} = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \\ 64 & 128 & 256 \end{bmatrix}$.

Calculate a basis for each of the following four subspaces:
$\mathcal{N}(\mathbf{B})$, $\mathcal{N}(\mathbf{B}^\mathsf{T})$, $\mathcal{N}(\mathbf{B})^\perp$ and $\mathcal{N}(\mathbf{B}^\mathsf{T})^\perp$ (the null spaces (kernels) of $\mathbf{B}$ and $\mathbf{B}^\mathsf{T}$, and the orthogonal complements of these two subspaces).
Also answer the following question: For which value on the constant b does the vector $(1, b, 1)^\mathsf{T}$ belong to $\mathcal{N}(\mathbf{B})$? .... (4p)

2. This exercise deals with the following LP problem on so called *general form*:

$$\begin{array}{lll} \text{P:} & \text{minimize} & x_3 \\ & \text{subject to} & x_1 - x_2 + x_3 \geq 0, \\ & & x_2 + x_3 \geq 0, \\ & & x_1 + x_2 = 3, \\ & & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \text{ "free" (not sign-restricted).} \end{array}$$

(a) Transform this problem to so called *standard form*, i.e. to a problem with equality constraints and sign-restricted (non-negative) variables.
(Hint: The free variable x_3 can be written as the difference between two sign-restricted variables.) .... (2p)

(b) Show that the following solution is optimal to the problem P:
$\hat{x}_1 = 2,\ \hat{x}_2 = 1,\ \hat{x}_3 = -1.$ .... (5p)

(c) Formulate the dual LP problem D corresponding to P,
and calculate an optimal solution to this dual problem D.
Also check that the optimal values of P and D are equal. .... (3p)

3. This exercise deals with quadratic optimization. Let

$$f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 3x_1x_2 - 3x_2x_3 - 3x_3x_1 + 10x_1 + 20x_2 + 30x_3,$$

where $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T} \in I\!R^3$.

(a) Is there any globally optimal solution to the problem of minimizing $f(\mathbf{x})$ without any constraints? In that case, calculate such a solution. .... (4p)

(b) Use a null space method to decide whether there is any globally optimal solution to the problem of minimizing $f(\mathbf{x})$ subject to the constraint $x_1 + x_2 + x_3 = 3$. In that case, calculate such a solution. .... (6p)

4. This exercise deals with minimization of a certain non-linear function, namely the function

$$f(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + 1} \; - \; 0.3\,x_1 \; - \; 0.4\,x_2\,,$$

where $\mathbf{x} = (x_1, x_2)^\mathsf{T} \in I\!R^2$.

(a) Use $\mathbf{x}^{(1)} = (0, 0)^\mathsf{T}$ as a starting point, and calculate the next iteration point $\mathbf{x}^{(2)}$ with Newton's method. .... (4p)

(b) Is f a convex function on the whole set $I\!R^2$? Motivate your answer. ..... (2p)

(c) Calculate (analytically) a globally optimal solution to the problem of minimizing $f(\mathbf{x})$ without any constraints. .... (4p)

5. This exercise deals with Lagrange relaxation and dual problems.
Consider the following "primal" problem P in the variables x_j:

$$\text{P:} \quad \text{minimize} \quad \sum_{j=1}^{n} \frac{c_j}{1-x_j}$$
$$\text{subject to} \quad \sum_{j=1}^{n} \frac{1}{1+x_j} \leq n,$$
$$-1 < x_j < 1 \text{ for } j = 1, \ldots, n,$$

where $n > 1$ is a given integer and $c_j > 0$ are given strictly positive real numbers.

Let the constraint $\displaystyle\sum_{j=1}^{n} \frac{1}{1+x_j} - n \leq 0$ be the only explicit constraint ($g(\mathbf{x}) \leq 0$), while the constraints $-1 < x_j < 1$ are considered to be implicit constraints ($\mathbf{x} \in X$).

(a) Use Lagrange relaxation (with respect to the explicit constraint) to deduce an *explicit* expression for the dual objective function $\varphi(y)$, valid for all $y > 0$. (3p)

Assume from now on that $n = 3$, $c_1 = 1$, $c_2 = 4$ and $c_3 = 9$.

(b) Calculate a number $\hat{y} > 0$ such that $\varphi(\hat{y}) \geq \varphi(y)$ for all $y > 0$. .... (3p)

(c) Is the above number $\hat{y}$ unique? Motivate! .... (1p)

(d) Calculate the corresponding solution $\hat{\mathbf{x}}$ to the primal problem P, and show that this is an optimal solution to P. .... (3p)

(e) Is your calculated $\hat{\mathbf{x}}$ the only optimal solution to P? Motivate! .... (1p)

Good luck!