KTH Matematik

Exam in SF1811/SF1841 Optimization.
January 14, 14:00–19:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2014. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

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1. A certain small network has two source nodes, called node 1 and node 2, with given supplies 300 units (for node 1) and 600 units (for node 2), and two sink nodes, called node 3 and node 4, with given demands 400 units (for node 3) and 500 units (for node 4). There are directed links (arcs) from each of the source nodes to each of the sink nodes, i.e. four links (1,3), (1,4), (2,3) and (2,4), with corresponding costs c_{13}, c_{14}, c_{23} and c_{24} per unit flow in the link.

(a) The corresponding minimum cost flow problem can be formulated as an LP problem on the form: minimize $\mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
Write down, in details, the matrix $\mathbf{A}$ and the vectors $\mathbf{b}$ and $\mathbf{c}$ for this example, and describe what the components in the variable vector $\mathbf{x}$ stand for. . . . (2p)

(b) Since the total supply equals the total demand, it is a *balanced* network flow problem, and then each *spanning tree* in the network (disregarding the directions of the links) correspond to a *basic solution* of the LP problem.
In the current example there are four different spanning trees (each containing three links). Draw these spanning trees in four figures. Then calculate and write down the corresponding flows in the links. As you notice, only two of these four basic solutions are *feasible* basic solutions. Which two? . . . .(3p)

(c) First assume that $c_{13} - c_{14} - c_{23} + c_{24} > 0$.
What is the optimal solution to the problem for this case?
Next assume that $c_{13} - c_{14} - c_{23} + c_{24} < 0$.
What is the optimal solution to the problem for this case?
Finally assume that $c_{13} - c_{14} - c_{23} + c_{24} = 0$.
Calculate *three different* optimal solutions to the problem for this case. . (3p)

2. Consider the following LP problem on standard form:

$$\begin{array}{rl} \text{minimize} & 4x_1 + 4x_2 + 2x_3 + 4x_4 + 4x_5 \\ \text{subject to} & x_2 + 2x_3 + 3x_4 + 4x_5 = 8, \\ & 4x_1 + 3x_2 + 2x_3 + x_4 = 4, \\ & x_j \geq 0, \;\; j = 1, \ldots, 5. \end{array}$$

(a) Use the Simplex method to calculate an optimal solution to the problem. Start with x_1 and x_5 as basic variables. . . . . (5p)

(b) Formulate the corresponding dual LP problem, illustrate it in a figure, and give an optimal solution to it. . . . . (3p)

(c) Assume that the second constraint in the above problem is removed, so that the primal problem instead becomes:

$$\begin{array}{rl} \text{minimize} & 4x_1 + 4x_2 + 2x_3 + 4x_4 + 4x_5 \\ \text{subject to} & x_2 + 2x_3 + 3x_4 + 4x_5 = 8, \\ & x_j \geq 0, \;\; j = 1, \ldots, 5. \end{array}$$

Is the optimal solution you obtained in (a) above an optimal solution also to this problem? Motivate! . . . . (2p)

(d) Now assume that the first (and not the second) constraint in the original problem is removed, so that the primal problem becomes:

$$\begin{array}{rl} \text{minimize} & 4x_1 + 4x_2 + 2x_3 + 4x_4 + 4x_5 \\ \text{subject to} & 4x_1 + 3x_2 + 2x_3 + x_4 = 4, \\ & x_j \geq 0, \;\; j = 1, \ldots, 5. \end{array}$$

Is the optimal solution you obtained in (a) above an optimal solution also to this problem? Motivate! . . . . (2p)

3. This exercise deals with the quadratic function

$$f(\mathbf{x}) = \tfrac{1}{2}\,\mathbf{x}^\mathsf{T}\mathbf{H}\mathbf{x}, \;\; \text{with } \mathbf{H} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 1 \end{bmatrix},$$

where $\mathbf{x} = (x_1, x_2, x_3)^\mathsf{T} \in I\!R^3$ is the variable vector.
You shall investigate if the function can be minimized over a given feasible $\mathcal{F}$, for three different choices of $\mathcal{F}$.
More precisely, you should answer the following question for each of the three cases (a), (b) and (c) below:

"Is there any vector $\hat{\mathbf{x}} \in \mathcal{F}$ such that $f(\hat{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{F}$?"

Whenever your answer is "yes", you should calculate *all* such minimizing vectors $\hat{\mathbf{x}}$.
Motivate your answers carefully!

(a) $\mathcal{F} = I\!R^3$ (i.e. no constraints at all). . . . . (3p)

(b) $\mathcal{F} = \{\mathbf{x} \in I\!R^3 \mid x_1 - x_2 + x_3 = 0\}$ (i.e. one constraint). . . . . (3p)

(c) $\mathcal{F} = \{\mathbf{x} \in I\!R^3 \mid x_1 + x_2 = 0 \text{ and } x_2 + x_3 = 0\}$ (i.e. two constraints). . . . . (3p)

4. Consider the following problem P with linear inequality constraints:

$$\text{P:} \quad \text{minimize} \quad \tfrac{1}{2}\,\|\,\mathbf{x}-\mathbf{q}\,\|^2 = \tfrac{1}{2}\,(\mathbf{x}-\mathbf{q})^{\mathsf{T}}(\mathbf{x}-\mathbf{q})$$
$$\text{subject to} \quad \mathbf{A}\mathbf{x} \geq \mathbf{b},$$

where $\mathbf{A}$ is a given $m \times n$ matrix, $\mathbf{b} \in I\!R^m$ and $\mathbf{q} \in I\!R^n$ are given vectors, and $\mathbf{x} \in I\!R^n$ is the variable vector.

(a) Use *Lagrange relaxation* to deduce an *explicit* expression (containing $\mathbf{A}$, $\mathbf{b}$, $\mathbf{q}$ and $\mathbf{y}$) for the dual objective function $\varphi(\mathbf{y})$, valid for all dual variable vectors $\mathbf{y} \in I\!R^m$ with $\mathbf{y} \geq \mathbf{0}$. .... (3p)

From now on, in (b), (c) and (d) below, it is assumed that

$$n=4,\ m=2,\ \mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix},\ \mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix},\ \mathbf{q} = (1,2,2,1)^{\mathsf{T}}.$$

Note: It is possible to solve (b), (c) and (d) without solving (a).

(b) Give an explicit expression (containing the components y_1 and y_2 of the dual variable vector $\mathbf{y} \geq \mathbf{0}$) for the dual objective function $\varphi(\mathbf{y})$.
Then calculate a vector $\hat{\mathbf{y}} \geq \mathbf{0}$ such that $\varphi(\hat{\mathbf{y}}) \geq \varphi(\mathbf{y})$ for all $\mathbf{y} \geq \mathbf{0}$. ..... (3p)

(c) Calculate the corresponding solution $\hat{\mathbf{x}} \in I\!R^4$ to the primal problem P, and show that this is an optimal solution to P. .... (2p)

(d) Formulate the KKT conditions for the problem and verify that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ (from (b) and (c) above) satisfy these conditions. .... (3p)

5. Assume that m points $(a_i,\, b_i)$, $i = 1,\dots,m$, in the plane are given. Further asssume that these points lay “almost”, but not exactly, on a circle. This exercise deals with how to find the, in some sense, “best” circle for the given points.

(a) One reasonable formulation of this problem is as the following nonlinear least-squares problem in the variables x, y and r :

$$\text{minimize} \quad f(x,y,r) = \tfrac{1}{2}\sum_{i=1}^{m}\left(\sqrt{(x-a_i)^2+(y-b_i)^2} - r\,\right)^2$$

where r is the radius of the circle C we search for, while (x,y) is the location of the center of C.
Assume that $m = 4$ and that the given points are $(5,0)$, $(0,6)$, $(-4,0)$, $(0,-5)$.
Start with the guess that $(x,y,r) = (0,0,5)$ and carry out one complete iteration with *Gauss-Newtons method*.
What are the new values of x, y and r after this iteration? .... (5p)

(b) An alternative approach is to search for two circles, C_1 and C_2, such that:

- C_1 and C_2 have the same locations of their centers, but the radius of C_2 is larger than the radius of C_1.
- Each of the given points (a_i, b_i) is either *on* one of the circles or strictly *between* the two circles (i.e. inside C_2 and outside C_1).
- The area between the two circles is as small as possible, subject to the above requirements.

Formulate this problem as a nonlinear optimization problem with inequality constraints, where both the objective function and all the constraint functions are continuously differentiable (i.e. their partial derivatives exist and are continuous everywhere). You should *not* calculate any numerical solution. . . (3p)

(c) The problem in (b) can in fact be formulated as an LP problem!
Try to find such a formulation. . . . . (2p)

Good luck!