KTH Matematik

Exam in SF1811 Optimization.
January 13, 2016, 14:00–19:00.

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Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2015. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different exercises 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

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1. This problem deals with a certain network with four nodes and four *directed* arcs. The nodes 1 and 2 are source nodes with given supplies 120 and 80, while the nodes 3 and 4 are sink nodes with given demands 40 and 160.
The equations for the flow balances in the four nodes can be written on the form $\mathbf{Ax} = \mathbf{b}$, where the incidence matrix $\mathbf{A}$, the variable vector $\mathbf{x}$ with link flows, and the right hand side vector $\mathbf{b}$ are:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_{12} \\ x_{13} \\ x_{24} \\ x_{34} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 120 \\ 80 \\ -40 \\ -160 \end{pmatrix}.$$

(a) Illustrate the network in a figure. .... (1p)

(b) Let $\mathbf{c} = (c_{12}, c_{13}, c_{24}, c_{34})^\mathsf{T}$ be the vector of costs per unit flow in the different arcs, and assume that $\mathbf{c} = (6,\, 4,\, 2,\, 5)^\mathsf{T}$. Use the simplex method for minimum cost network flow problems to calculate an optimal solution to the problem: minimize $\mathbf{c}^\mathsf{T}\mathbf{x}$ subject to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.
Start with the three variables x_{13}, x_{24} and x_{34} as basic variables. .... (5p)

(c) Now assume that instead of the linear objective function used in (b), there is a quadratic objective function $\frac{1}{2}(x_{12}^2 + x_{13}^2 + x_{24}^2 + x_{34}^2)$.
Further assume that the variables x_{ij} are no longer required to be non-negative. Then the problem becomes a QP problem with linear equality constraints.
Use a *nullspace method* to solve this QP problem. .... (5p)

2. Consider the following LP problem on standard form:

$$\begin{aligned} \text{P:} \quad \text{minimize} \quad & 3x_1 + cx_2 + cx_3 - x_4 \\ \text{subject to} \quad & x_1 + x_2 - x_3 - x_4 = 3, \\ & x_1 - x_2 + x_3 - x_4 = 7, \\ & x_j \geq 0, \ j = 1, 2, 3, 4, \end{aligned}$$

where c is given real constant.

Note: (b), (c), (d) and (e) below can be solved without solving (a).

(a) For this problem, it is easy to calculate all the *basic feasible solutions*. Do that! .... (2p)

(b) Show that if $c = 1$ then $\mathbf{x} = (5,\ 0,\ 2,\ 0)^\mathsf{T}$ is the unique optimal solution. . (2p)

(c) Show that if $c = -1$ then there is no optimal solution. Further, for this case, calculate a feasible solution with objective value less than -1000. .... (2p)

(d) Show that if $c = 0$ then there is an infinite number of optimal solutions. Further, write down three different optimal solutions for this case. ...... (2p)

(e) Formulate the dual LP problem to the above problem P. For each of the above three cases, i.e. for $c = 1$, $c = -1$ and $c = 0$, illustrate the corresponding dual problem in a figure, and calculate graphically an optimal dual solution (when it exists). .... (3p)

3. Let the nonlinear function $f : I\!R^2 \to I\!R$ be defined by

$$f(\mathbf{x}) = (x_1 - x_2)^3 + (x_1 - x_2)^2 + (x_2 - 1)^2 \ \text{ where } \ \mathbf{x} = (x_1, x_2)^\mathsf{T}.$$

In this exercise we consider the problem of minimizing $f(\mathbf{x})$ without any constraints.

(a) Perform a complete iterations with *Newtons method*, starting from the point $\mathbf{x}^{(1)} = (0,\ 0)^\mathsf{T}$. .... (6p)

(b) Calculate (analytically) all the local optimal solutions to the problem. .. (2p)

(c) Is there any global optimal solution to the problem? .... (1p)

(d) Find two real number a_1 and a_2 such that the above function f is convex on the convex set $C = \{\, (x_1, x_2)^\mathsf{T} \in I\!R^2 \mid a_1x_1 + a_2x_2 \leq 1\}$. .... (2p)

4. Consider the following QP problem with linear *inequality* constraints:

$$\begin{aligned} \mathrm{P}: \quad & \text{minimize} && \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{x} \\ & \text{subject to} && \mathbf{A}\mathbf{x} \geq \mathbf{b}, \end{aligned}$$

where $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{pmatrix} 18 \\ 30 \end{pmatrix}$ and $\mathbf{x} = (x_1, x_2, x_3, x_4)^{\mathsf{T}}$.

(a) Use *Lagrange relaxation* to deduce an *explicit* expression for the dual objective function $\varphi(\mathbf{y})$, valid for all dual variable vectors $\mathbf{y} \in I\!R^2$ with $\mathbf{y} \geq \mathbf{0}$. .... (3p)

(b) Show that there is a dual vector $\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2)^{\mathsf{T}}$ with $\hat{y}_1 = 0$ and $\hat{y}_2 > 0$, and a primal vector $\hat{\mathbf{x}} \in I\!R^4$, such that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ together satisfy the global optimality conditions (GOC). .... (6p)

5. Let f be a quadratic function defined by

$$f(\mathbf{x}) = \tfrac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{H}\mathbf{x} + \mathbf{c}^{\mathsf{T}}\mathbf{x},$$

where $\mathbf{H}$ is a given symmetric $n \times n$ matrix,
$\mathbf{c} \in I\!R^n$ is a given vector, and
$\mathbf{x} \in I\!R^n$ is the variable vector.

In this exercise (in both (a) and (b) below), it is assumed that:

- The given vector $\mathbf{c}$ is *not* the zero vector (i.e. $\mathbf{c} \neq \mathbf{0}$).
- The given matrix $\mathbf{H}$ is positive semidefinite, but *not* positive definite.
- There is a vector $\hat{\mathbf{x}} \in I\!R^n$ which satisfies that $f(\hat{\mathbf{x}}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in I\!R^n$.

(a) Prove that $\hat{\mathbf{x}} \neq \mathbf{0}$, $f(\hat{\mathbf{x}}) < 0$ and $\hat{\mathbf{x}}^{\mathsf{T}}\mathbf{H}\hat{\mathbf{x}} > 0$. .... (4p)

(b) Let $\hat{\mathbf{x}}$ be as above, and let $k \in I\!R$ be a constant.
Then consider the following problem:

$$\begin{aligned} \mathrm{P}_1: \quad & \text{minimize} && f(\mathbf{x}) \\ & \text{subject to} && \mathbf{c}^{\mathsf{T}}\mathbf{x} = k\,\mathbf{c}^{\mathsf{T}}\hat{\mathbf{x}}. \end{aligned}$$

For which values on k is the optimal value to this problem > 0? .... (4p)

Good luck!