KTH Matematik

Exam in SF1811 Optimization.
January 11, 2017, 14:00–19:00.

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2016. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different exercises 1,2,3,4,5 on separate sheets.
This is important since the exams are split up during the corrections.

---

1. A certain network with six nodes and eight directed arcs has two source nodes, called node 1 and node 2, with given supplies 40 units (for node 1) and 30 units (for node 2), two intermediate nodes, called node 3 and node 4, without any supplies or demands, and two sink nodes, called node 5 and node 6, with given demands 20 units (for node 5) and 50 units (for node 6). There are directed arcs from each source node to each intermediate nodes, and also from each intermediate node to each sink node. Thus, the set of arcs is $\mathcal{A} = \{ (1,3), (1,4), (2,3), (2,4), (3,5), (3,6), (4,5), (4,6) \}$, where (i,j) denotes the arc from node i to node j. The vector with costs per unit flow in the arcs is given by $\mathbf{c} = (c_{13}, c_{14}, c_{23}, c_{24}, c_{35}, c_{36}, c_{45}, c_{46})^{\mathsf{T}} = (2,2,2,4,3,4,3,3)^{\mathsf{T}}$, where c_{ij} denotes the cost per unit flow in the arc (i,j).

We are interested in finding an optimal solution to the *minimum cost flow problem* corresponding to the above network.

(a) An experienced planner suggests that flow should be sent only in the arcs $\{ (1,4), (2,3), (3,5), (3,6), (4,6) \}$, which correspond to a spanning tree in the network, while no flow should be sent in the arcs $\{ (1,3), (2,4), (4,5) \}$. Verify that the planner's suggestion indeed leads to an optimal solution. (4p)

(b) Since the above minimum cost flow problem is in fact an LP problem, there is a corresponding *dual* LP problem.
Formulate this dual LP problem in details (with the objective function and all the constraints explicitly expressed in the variables) and calculate an optimal solution to this dual LP problem. Your calculations in the above exercise (a) may of course be used. Verify that your dual solution is feasible to the dual problem, and that the primal and dual objective values are equal. . . . . (4p)

2. Consider the following LP problem P1 on standard form:

$$\begin{array}{lrl} \text{P1:} & \text{minimize} & 3x_1 + 3x_2 + 6x_3 \\ & \text{subject to} & 2x_1 + \ x_2 + 2x_3 = 6, \\ & & x_1 + \ x_2 + 4x_3 = 9, \\ & & x_j \geq 0,\ j = 1,2,3. \end{array}$$

(a) Verify that a BFS (*basic feasible solution*) is obtained by chosing x_2 and x_3 as basic variables.
Then start from this BFS and solve P1 with the simplex method. .... (5p)

(b) Assume that the right hand side of the first constraints in the above problem P1 is changed from 6 to 4, so that the following problem P2 is obtained:

$$\begin{array}{lrl} \text{P2:} & \text{minimize} & 3x_1 + 3x_2 + 6x_3 \\ & \text{subject to} & 2x_1 + \ x_2 + 2x_3 = 4, \\ & & x_1 + \ x_2 + 4x_3 = 9, \\ & & x_j \geq 0,\ j = 1,2,3. \end{array}$$

Verify that a BFS to P2 is *not* obtained with x_2 and x_3 as basic variables. (1p)

(c) The general method to search for a BFS to P2 is then to generate and solve the following so called "Phase 1 problem" in the original variables x_j and two *artificial* variables v_i :

$$\begin{array}{lrl} \text{P3:} & \text{minimize} & v_1 + v_2 \\ & \text{subject to} & 2x_1 + \ x_2 + 2x_3 + \ v_1 = 4, \\ & & x_1 + \ x_2 + 4x_3 + \ v_2 = 9, \\ & & x_j \geq 0,\ j = 1,2,3, \\ & & v_i \geq 0,\ i = 1,2. \end{array}$$

Verify that a BFS to P3 is obtained with v_1 and v_2 as basic variables.

Then start from this BFS and solve P3 with the simplex method.

The result should *either* be a BFS to the problem P2 in exercise (b),
or a conclusion that there are no feasible solutions at all to P2.

If you find a BFS to P2, you should carefully verify that this is the case.
If you conclude that there are no feasible solutions at all to P2, this conlcusion should be carefully motivated. .... (5p)

3. The four sides in a certain square are called *South*, *West*, *North* and *East*, while the four corners are called *SouthWest*, *NorthWest*, *NorthEast* and *SouthEast*.

Corresponding to each side, there is a given known number (e.g. the average of measured values of some physical quantity along the side).

These given side numbers are denoted, respectively, b_s, b_w, b_n and b_e.

In this exercise you should calculate numbers x_{sw}, x_{nw}, x_{ne} and x_{se} corresponding to the four corners (e.g. the unknown value of the above physical quantity in the corners) according to certain requirements described below.

First of all, these corner numbers should satisfy that each *side number* is the average of the two *corner numbers* corresponding to the side, i.e. that

$$\frac{x_{sw}+x_{se}}{2}=b_s\,,\quad \frac{x_{sw}+x_{nw}}{2}=b_w\,,\quad \frac{x_{nw}+x_{ne}}{2}=b_n\,,\quad \frac{x_{ne}+x_{se}}{2}=b_e\,. \tag{1}$$

(a) Show that it is possible find corner numbers which satisfy the constraints (1) if and only if the given side numbers satisfies that $\mathbf{w}^\mathsf{T}\mathbf{b}=0$, where $\mathbf{w}=(1,-1,\ 1,-1)^\mathsf{T}$ and $\mathbf{b}=(b_s,\, b_w,\, b_n,\, b_e)^\mathsf{T}$.
Also show that if $\mathbf{w}^\mathsf{T}\mathbf{b}=0$ then there is an infinite number of vectors $\mathbf{x}=(x_{sw},\, x_{nw},\, x_{ne},\, x_{se})^\mathsf{T}$ which satisfy the above constraints (1). .... (3p)

(b) This exercise (b) can be solved even if you failed with the proofs in (a).
Assume that $\mathbf{b}=(b_s,\, b_w,\, b_n,\, b_e)^\mathsf{T}$ satisfies $\mathbf{w}^\mathsf{T}\mathbf{b}=0$.
Then, in addition to the above constraints (1), one would like that each corner number should be the average of the side numbers for the two sides who meet in the corner, i.e that

$$x_{sw}=b_{sw},\quad x_{nw}=b_{nw},\quad x_{ne}=b_{ne},\quad x_{se}=b_{se}, \tag{2}$$

where the numbers in the right hand sides are given by

$$b_{sw}=\frac{b_s+b_w}{2},\quad b_{nw}=\frac{b_n+b_w}{2},\quad b_{ne}=\frac{b_n+b_e}{2},\quad b_{se}=\frac{b_s+b_e}{2}.$$

But since it is not possible in general to satisfy simultaneously both (1) and (2), one would instead like to find corner numbers which are optimal to the following quadratic optimization problem:

$$\begin{array}{l}\text{minimize } \ \frac{1}{2}\left((x_{sw}-b_{sw})^2+(x_{nw}-b_{nw})^2+(x_{ne}-b_{ne})^2+(x_{se}-b_{se})^2\right)\\ \text{subject to that } \mathbf{x}=(x_{sw},\, x_{nw},\, x_{ne},\, x_{se})^\mathsf{T} \text{ satisfies the constraints (1).}\end{array} \tag{3}$$

Assume that $\mathbf{b}=(b_s,\, b_w,\, b_n,\, b_e)^\mathsf{T}=(1,\,3,\,4,\,2)^\mathsf{T}$, which satisfies $\mathbf{w}^\mathsf{T}\mathbf{b}=0$.
Then solve the above quadratic optimization problem (3) using a nullspace method. What are the optimal corner numbers? .... (6p)

4. Consider the following problem P in the variables x_1, x_2 and x_3,

$$\begin{array}{lll} \text{P:} & \text{minimize} & \dfrac{1}{x_1}+\dfrac{1}{x_2}+\dfrac{1}{x_3}+9x_1+16x_2+25x_3 \\ & \text{subject to} & \dfrac{1}{x_1}+\dfrac{1}{x_2}+\dfrac{1}{x_3} \le b, \\ & & x_j > 0, \text{ for } j=1,2,3, \end{array}$$

where $b > 0$ is a given strictly positive real number.

Let the constraint $\dfrac{1}{x_1}+\dfrac{1}{x_2}+\dfrac{1}{x_3}-b \le 0$ be the only explicit constraint ($g(\mathbf{x}) \le 0$), while the constraints $x_j > 0$ are considered to be implicit constraints ($\mathbf{x} \in X$).

(a) Use Lagrange relaxation (with respect to the explicit constraint) to deduce an *explicit* expression for the dual objective function $\varphi(y)$, valid for $y \ge 0$. (4p)

(b) Assume that $b = 6$.
Calculate a number $\hat{y} \ge 0$ such that $\varphi(\hat{y}) \ge \varphi(y)$ for all $y \ge 0$.
Then calculate a vector $\hat{\mathbf{x}} \in X$ which together with $\hat{y}$ satisfies the global optimality condition (GOC) for P.
Finally, check that the primal objective value for $\mathbf{x} = \hat{\mathbf{x}}$ is equal to the dual objective value for $y = \hat{y}$. .... (3p)

(c) Repeat all the steps in the above exercise (b), but now with $b = 18$. .... (3p)

5. Let the nonlinear function $f : I\!R^2 \to I\!R$ be defined by

$$f(\mathbf{x}) = x_1^2x_2^2 + 2x_1^2 + 2x_2^2 - 12x_1 - 12x_2, \quad \text{where } \mathbf{x} = (x_1, x_2)^{\mathsf{T}}.$$

(a) Perform a complete iteration with *Newtons method* for minimizing $f(\mathbf{x})$ without any constraint, starting from the point $\mathbf{x}^{(1)} = (1,\, 0)^{\mathsf{T}}$.
Make sure that $f(\mathbf{x}^{(2)}) < f(\mathbf{x}^{(1)})$. .... (5p)

(b) Decide, for each of the following three convex sets, if the above function f is a *convex function on the set* or not.
$C_1 = \{\mathbf{x} \in I\!R^2 \mid \mathbf{x}^{\mathsf{T}}\mathbf{x} < 1\}$, $C_2 = \{\mathbf{x} \in I\!R^2 \mid \mathbf{x}^{\mathsf{T}}\mathbf{x} < 4\}$, $C_3 = \{\mathbf{x} \in I\!R^2 \mid \mathbf{x}^{\mathsf{T}}\mathbf{x} < 9\}$.
Motivate your answers. .... (3p)

(c) Is there any point $\tilde{\mathbf{x}} \in I\!R^2$ such that $\tilde{x}_1 \ne \tilde{x}_2$ *and* $\tilde{\mathbf{x}}$ is a local optimal solutions to the problem of minimizing $f(\mathbf{x})$ without any constraints?
Motivate your answer. .... (2p)

(d) Is there any point $\tilde{\mathbf{x}} \in I\!R^2$ such that $\tilde{x}_1 = \tilde{x}_2$ *and* $\tilde{\mathbf{x}}$ is a local optimal solutions to the problem of minimizing $f(\mathbf{x})$ without any constraints?
Motivate your answer. .... (2p)

Good luck!