KTH Matematik

Exam in SF1811/SF1841 Optimization.
Friday, March 14, 8:00–13:00

Examiner: Krister Svanberg, telephone: 790 7137, email: krille@math.kth.se.

Allowed utensils: Pen, paper, eraser and ruler. (Penna, papper, suddgummi och linjal.)
No calculator! (Ingen räknare!) A formula-sheet is handed out.

Language: Your solutions should be written in English or in Swedish.

Unless otherwise stated in the problem statement, the problems should be solved using systematic methods that do not become unrealistic for large problems. Unless otherwise stated in the problem statement, known theorems can be used without proving them, as long as they are formulated correctly. Motivate all your conclusions carefully.
A passing grade E is guarranteed for 25 points, including bonus points from the home assignments during Nov-Dec 2013. 23-24 points give a possibility to complement the exam to grade E within three weeks from the announcement of the results. Contact the examiner as soon as possible by email if this is the case.
Write your name on each page of the solutions you hand in and number the pages.
Write the solutions to the different questions 1,2,3,4,5 on separate sheets.
(This is important since the exams are split up during the corrections.)

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1. The LP problem below in the variables x_{ik} and z_{kj} is in fact a minimum cost network flow problem with two supply nodes ($i = 1, 2$), two transshipment nodes ($k = 1, 2$), and 2 demand nodes ($j = 1, 2$). There is an arc from each supply node i to each transshipment node k, with flow variable x_{ik}, and from each transshipment node k to each demand node j, with flow variable z_{kj}.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{2}\sum_{k=1}^{2} p_{ik}x_{ik} + \sum_{k=1}^{2}\sum_{j=1}^{2} q_{kj}z_{kj} \\
\text{subject to} \quad & x_{11} + x_{12} = 30, \\
& x_{21} + x_{22} = 20, \\
& -x_{11} - x_{21} + z_{11} + z_{12} = 0, \\
& -x_{12} - x_{22} + z_{21} + z_{22} = 0, \\
& -z_{11} - z_{21} = -40, \\
& -z_{12} - z_{22} = -10, \\
& x_{ik} \geq 0\,,\ z_{kj} \geq 0, \quad \text{for all } i, k, j.
\end{aligned}
$$

Assume that the following cost data are given:
$p_{11} = 5$, $p_{12} = 2$, $p_{21} = 3$, $p_{22} = 2$, $q_{11} = 5$, $q_{12} = 5$, $q_{21} = 7$, $q_{22} = 6$.

(a) The following solution is suggested by someone:
$x_{11} = 0$, $x_{12} = 30$, $x_{21} = 20$, $x_{22} = 0$, $z_{11} = 10$, $z_{12} = 10$, $z_{21} = 30$, $z_{22} = 0$.
Show that this is a feasible basic solution to the problem. .... (4p)

(b) Calculate an optimal basic solution to the problem. .... (5p)

2. Assume that the following five vectors in $\mathbb{R}^3$ are given:

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{a}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

Let $\mathbf{A}$ be a 3×4 matrix with the above four vectors $\mathbf{a}_j$ as columns.
One wants to find out whether there are *non-negative* scalars (real numbers) x_1, x_2, x_3 and x_4 such that $\mathbf{b} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \mathbf{a}_4 x_4$, i.e. $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$.

Therefore, the following LP problem in the seven non-negative variables $\mathbf{x} = (x_1,\, x_2,\, x_3,\, x_4)^\mathsf{T}$ and $\mathbf{v} = (v_1,\, v_2,\, v_3)^\mathsf{T}$ is considered:

$$\begin{aligned} \text{minimize} \quad & \mathbf{e}^\mathsf{T}\mathbf{v} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}\,\mathbf{v} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \ \text{ and } \ \mathbf{v} \geq \mathbf{0}, \end{aligned}$$

where $\mathbf{I}$ is a 3×3 unit matrix, and $\mathbf{e} = (1,\, 1,\, 1)^\mathsf{T}$.
(Since $\mathbf{b} \geq \mathbf{0}$, this problem has the obvious feasible basic solution $\mathbf{x}=\mathbf{0}$, $\mathbf{v}=\mathbf{b}$.)

(a) Show that an optimal solution to the above LP problem is given by $\mathbf{x} = (2,\, 5,\, 0,\, 0)^\mathsf{T}$ and $\mathbf{v} = (0,\, 0,\, 1)^\mathsf{T}$. .... (5p)

(b) Are there any scalars $x_j \geq 0$ such that $\mathbf{b} = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \mathbf{a}_4 x_4$?
Motivate your answer carefully. .... (2p)

(c) Is there any vector $\mathbf{y} \in \mathbb{R}^3$ such that $\mathbf{b}^\mathsf{T}\mathbf{y} > 0$ and $\mathbf{a}_j^\mathsf{T}\mathbf{y} \leq 0$ for all j,
i.e. $\mathbf{b}^\mathsf{T}\mathbf{y} > 0$ and $\mathbf{A}^\mathsf{T}\mathbf{y} \leq \mathbf{0}$? If "yes", calculate such a vector $\mathbf{y}$. .... (3p)
(*Hint:* Consider the dual problem corresponding to the above LP problem.)

3. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$ and $\mathbf{q} = (q_1, q_2, q_3, q_4)^\mathsf{T} \in \mathbb{R}^4$ be given.

(a) Assume first that one wants to find the vector $\hat{\mathbf{x}}$ in the *null space* $\mathcal{N}(\mathbf{A})$ of $\mathbf{A}$ which is closest to $\mathbf{q}$ among all vectors $\mathbf{x}$ in $\mathcal{N}(\mathbf{A})$.
Thus, one wants to solve the problem

$$\text{minimize } \tfrac{1}{2}\|\mathbf{x} - \mathbf{q}\|^2 \ \text{ subject to } \mathbf{A}\mathbf{x} = \mathbf{0},$$

where, as usual, $\frac{1}{2}\|\mathbf{x} - \mathbf{q}\|^2 = \frac{1}{2}(\mathbf{x} - \mathbf{q})^\mathsf{T}(\mathbf{x} - \mathbf{q})$.
Calculate an optimal solution $\hat{\mathbf{x}} \in \mathbb{R}^4$, and a corresponding vector $\hat{\mathbf{u}} \in \mathbb{R}^2$ of Lagrange multipliers, to this problem.
Your answers should of course contain the components q_j of $\mathbf{q}$. .... (5p)

(b) Assume next that one wants to find the vector $\hat{\mathbf{y}}$ in the *column space* $\mathcal{R}(\mathbf{A}^\mathsf{T})$ of $\mathbf{A}^\mathsf{T}$ which is closest to $\mathbf{q}$ among all vectors $\mathbf{y}$ in $\mathcal{R}(\mathbf{A}^\mathsf{T})$.
Thus, one now wants to solve the problem

$$\text{minimize } \tfrac{1}{2}\|\mathbf{y} - \mathbf{q}\|^2 \ \text{ subject to } \mathbf{y} = \mathbf{A}^\mathsf{T}\mathbf{v} \text{ for some } \mathbf{v} \in \mathbb{R}^2.$$

Calculate an optimal solution $(\hat{\mathbf{y}}, \hat{\mathbf{v}})$ to this problem!
Your answers should of course contain the components q_j of $\mathbf{q}$. .... (5p)

4. Let $f(\mathbf{x}) = x_1^2 x_2^2 + x_1^2 + 3x_2^2 - 2x_1 x_2 - 4x_1 - 4x_2$, where $\mathbf{x} = (x_1, x_2)^\mathsf{T} \in I\!R^2$, and consider first the problem of minimizing $f(\mathbf{x})$ without any constraints.

(a) Use $\mathbf{x}^{(1)} = (0,0)^\mathsf{T}$ as the starting point, and calculate the next iteration point $\mathbf{x}^{(2)}$ by Newtons method. .... (4 p)

(b) Is $f(\mathbf{x})$ a convex function on the whole set $I\!R^2$? Motivate! .... (2 p)

(c) Assume now that the above function $f(\mathbf{x})$ should be minimized subject to the constraint $x_1 - x_2 = 0$.
Use any method you like to calculate a globally optimal solution $\hat{\mathbf{x}}$ to this problem. You must prove that your solution is globally optimal. .... (4 p)

5. Consider the following problem P:

$$
\begin{array}{lll}
\text{P:} & \text{minimize} & \displaystyle\sum_{i=1}^{2}\sum_{j=1}^{3}(x_{ij}\ln(x_{ij}) - x_{ij}) \\
& \text{subject to} & x_{11} + x_{12} + x_{13} \le 2/5, \\
& & x_{21} + x_{22} + x_{23} \le 3/5, \\
& & x_{11} + x_{21} \le 1/6, \\
& & x_{12} + x_{22} \le 1/3, \\
& & x_{13} + x_{23} \le 1/2, \\
& & x_{ij} > 0 \text{ for all } i, j,
\end{array}
$$

where $\ln(x_{ij})$ is the natural logarithm of x_{ij}.

Let the constraints "$x_{ij} > 0$ for all i, j" be the implicit constraints ($\mathbf{x} \in X$) and let the first five constraints be the explicit constraints with corresponding Lagrange multipliers (dual variables) $\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3$.

(a) Use Lagrange relaxation (with respect to the explicit constraints) to deduce an *explicit* expression for the dual objective function $\varphi(\lambda_1, \lambda_2, \mu_1, \mu_2, \mu_3)$, valid for all $\lambda_i \ge 0$ and $\mu_j \ge 0$, and formulate the dual problem D. ...... (3p)

(b) Show that an optimal solution to the dual problem D is given by
$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3) = (\ln(5/2), \ln(5/3), \ln(6), \ln(3), \ln(2))$,
and calculate an optimal solution $\hat{\mathbf{x}}$ to the primal problem P.
Also verify that the optimal values of P and D are equal in this example. (5p)

(c) Is the above optimal solution to D the only optimal solution to D, and is your calculated optimal solution to P the only optimal solution to P? Motivate your answers carefully. .... (3p)

Good luck!