Exercise 19.5.

(1) Let x_0 be a feasible point. Then $x_0^T P x_0 = 1$.
In particular $x_0 \neq 0$ (for otherwise $0^T P 0 = 0 \neq 1$!).
With

$$h(x) := x^T P x - 1 \qquad (x \in \mathbb{R}^n)$$

we have that the feasible set

$$\mathcal{F} = \{x \in \mathbb{R}^n : x^T P x = 1\} = \{x \in \mathbb{R}^n : h(x) = 0\}$$

We have $\nabla h(x) = 2x^T P$ $(x \in \mathbb{R}^n)$. Hence if $v \in \mathbb{R}^1$ is such that $v(\nabla h(x_0)) = 0$, then $v(2x_0^T P) = 0$ and since $x_0^T P \neq 0$, it follows that $v = 0$. Thus x_0 is a regular point.

(2) We have $f(x) = -x^T Q x$ and so $\nabla f(x) = -2x^T Q$.
So if $\hat{x}$ is an optimal solution, we have: $\exists\, \hat{u} \in \mathbb{R}$ such that:

$$\hat{x}^T P \hat{x} = 1, \qquad (*)$$

$$-2\hat{x}^T Q + \hat{u}(2\hat{x}^T P) = 0. \qquad (**)$$

By postmultiplying $(**)$ by $\hat{x}$ and using $(*)$, we get

$$-2\hat{x}^T Q \hat{x} + \hat{u}\, 2 \underbrace{\hat{x}^T P \hat{x}}_{=1} = -2\hat{x}^T Q \hat{x} + 2\hat{u} = 0$$

and so

$$\hat{u} = +\hat{x}^T Q \hat{x}.$$

Hence from $(**)$ we have

$$P^{-1} Q \hat{x} = \hat{u} \hat{x}.$$

Thus $\hat{x}$ is an eigenvector of $P^{-1}Q$ corresponding to the eigenvalue $\hat{u}$. (Note that since $\hat{x}$ satisfies $\hat{x}^T P \hat{x} = 1$, we know that $\hat{x} \neq 0$.)

(3) We have $\hat{x}^T Q \hat{x}$

$= \hat{u}$

$=$ eigenvalue of $P^{-1}Q$ corresponding to the eigenvector $\hat{u}$.

Exercise 19.6

We have $f(x,y) = -(x+y)$

$$h(x,y) = \left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 - 1.$$

$$\nabla h(x,y) = \begin{bmatrix} -\frac{2a^2}{x^3} & -\frac{2b^2}{y^3} \end{bmatrix} \neq 0 \text{ for all } (x,y) \in \mathcal{H}$$

and so every $(x,y) \in \mathcal{H}$ is a regular point.

$$\nabla f(x,y) = \begin{bmatrix} -1 & -1 \end{bmatrix}.$$

So if $(\hat{x}, \hat{y})$ is a maximizer, then $\exists \hat{u}$ s.t.

$$\begin{bmatrix} -\frac{2a^2}{\hat{x}^3} & -\frac{2b^2}{\hat{y}^3} \end{bmatrix} + \hat{u} \begin{bmatrix} -1 & -1 \end{bmatrix} = 0$$

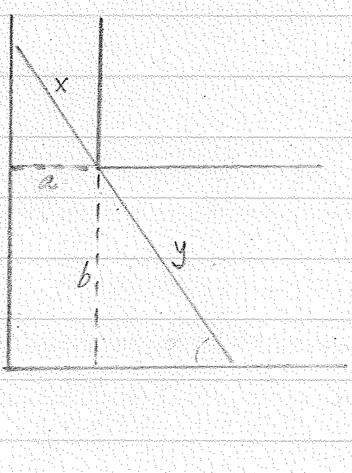
i.e, $\hat{u} = -\frac{2a^2}{\hat{x}^3} = -\frac{2b^2}{\hat{y}^3}$

and so $\hat{y} = \left(\frac{b^2}{a^2}\right)^{1/3} \hat{x}$.

But $\left(\frac{a}{\hat{x}}\right)^2 + \left(\frac{b}{\hat{y}}\right)^2 = 1$ and so $\frac{a^2}{\hat{x}^2} + \frac{b^2 a^{4/3}}{b^{4/3} \hat{x}^2} = 1$

i.e, $\hat{x} = a^{1/3} \sqrt{a^{2/3} + b^{2/3}}$.

Hence $\hat{y} = b^{1/3} \sqrt{a^{2/3} + b^{2/3}}$



By similarity of the two small triangles,

$$\frac{x}{a} = \frac{y}{\sqrt{y^2 - b^2}}$$

i.e., $\left(\frac{a}{x}\right)^2 = \frac{y^2 - b^2}{y^2} = 1 - \left(\frac{b}{y}\right)^2$

and so $\left(\frac{a}{x}\right)^2 + \left(\frac{b}{y}\right)^2 = 1$.

In light of this our solution implies that the largest possible length of the ladder (which can be carried through) is $\hat{x} + \hat{y} = \left(a^{2/3} + b^{2/3}\right)^{3/2}$.

Exercise 19.7

The problem is equivalent to

$$\text{minimize} \quad x^4 + y^4 + (1-x-y)^4$$
$$\text{subject to} \quad x^2 + y^2 + (1-x-y)^2 = 1.$$

We have

$$\nabla h(x,y) = [\, x-(1-x-y) \qquad y-(1-x-y)\,]$$

and if (x,y) is in the feasible set we have that $\nabla h(x,y)=0$ iff

$$(*) \quad \begin{cases} 2x+y=1 \\ x+2y=1 \\ x^2+y^2+(1-x-y)^2=1 \end{cases}$$

But $(*)$ has no solution. (Indeed the first two equations in $(*)$ give $x=y=1/3$ and then the third gives $\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=1$, which is false.) So every feasible solution is a regular point.

Let (x,y) be a local minimizer. Then $\exists u \in \mathbb{R}$ s.t

$$\nabla f(x,y) + u \nabla h(x,y) = 0$$

Hence

$$[4x^3 - 4(1-x-y)^3 \qquad 4y^3 - 4(1-x-y)^3] = -u[x-(1-x-y) \qquad y-(1-x-y)]$$

ie, $\exists \lambda \in \mathbb{R}$ s.t.

$$\left.\begin{aligned} (x-(1-x-y))(x^2+x(1-x-y)+(1-x-y)^2) &= \lambda(x-(1-x-y)) \\ (y-(1-x-y))(y^2+y(1-x-y)+(1-x-y)^2) &= \lambda(y-(1-x-y)). \end{aligned}\right\} (**)$$

1° $x-(1-x-y)=0$.

Then $y=1-2x$.

So $x^2+y^2+(1-x-y)^2=1$ gives $6x^2-4x=0$ i.e., $(3x-2)x=0$.

Hence $x=0$ or $x=2/3$. Correspondingly $y=1$ and $y=-1/3$, repectively.

So $(x,y)=(0,1)$ or $(\frac{2}{3},-\frac{1}{3})$.

$f(0,1) = 1^4+0^4+(1-1-0)^4 = 1$, while

$f(\frac{2}{3},-\frac{1}{3}) = \frac{16}{81}+\frac{1}{81}+\frac{16}{81} = \frac{33}{81} = \frac{11}{27} < 1 = f(0,1)$.

So $(\frac{2}{3},-\frac{1}{3})$ is a possible local minimizer.

2° $x-(1-x-y)\neq 0$

2.1° $y-(1-x-y)=0$.

Thus $x=1-2y$.

Similar to case 1° above we then get $(x,y)=(1,0)$ or $(-\frac{1}{3},\frac{2}{3})$.

Again $f(-\frac{1}{3},\frac{2}{3})=\frac{11}{27}\left(=f(\frac{2}{3},-\frac{1}{3})\right)$

Hence $(-\frac{1}{3},\frac{2}{3})$ is a possible local minimizer.

2.2° $y-(1-x-y)\neq 0$.

Then from $(**)$ we have

$$x^2+x(1-x-y)+(1-x-y)^2=y^2+y(1-x-y)+(1-x-y)^2$$

i.e., $(x-y)(x+y)=(y-x)(1-x-y)$. $(***)$

$2.2.1^\circ$ $x=y$.

Then $x^2+y^2+(1-x-y)^2=1$ gives $2x^2+(1-2x)^2=1$

i.e., $6x^2-4x=0$ and so $x=0$ or $\frac{2}{3}$.

Then $y=0$ and $y=2/3$, respectively.

So $(x,y)=(0,0)$ or $(\frac{2}{3},\frac{2}{3})$.

But $f(0,0)=1$ and $f(\frac{2}{3},\frac{2}{3})=\frac{11}{27}$

$\left(=f(-\frac{1}{3},\frac{2}{3})=f(\frac{2}{3},-\frac{1}{3})\right)$.

So $(\frac{2}{3},\frac{2}{3})$ is a possible local minimizer.

$2.2.2^\circ$ $x\neq y$. Then from $(***)$,

$x+y=-1+(x+y)$ and so $0=-1$, a contradiction.

So in light of all the above cases we know that if at all there is a local minimizer, then it has to be (each of) the points $(-\frac{1}{3},\frac{2}{3}),(\frac{2}{3},-\frac{1}{3}),(\frac{2}{3},\frac{2}{3})$.

But we know that a minimizer exists, since f is continuous and $\mathcal{K}=\{(x,y): x^2+y^2+(1-x-y)^2=1\}$ is compact. ($\mathcal{K}$ is an ellipse.)

Hence the points $(-\frac{1}{3},\frac{2}{3}),(\frac{2}{3},-\frac{1}{3}),(\frac{2}{3},\frac{2}{3})$ are global minimizer to our problem.

Correspondingly, $(-\frac{1}{3},\frac{2}{3},\frac{2}{3}),(\frac{2}{3},-\frac{1}{3},\frac{2}{3}),(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$ are global minimizers to the original optimization problem in the variables $(x,y,z)\in\mathbb{R}^3$.

Exercise 19.8.

(1) $h(x) = 0$ iff $0 \le x \le 1$ and so the feasible set

$$\mathcal{F} = \{x \in \mathbb{R} : h(x) = 0\} = [0,1].$$

(2) The problem reduces to:

$$\text{minimize } x$$
$$\text{subject to } x \in [0,1]$$

and so obviously $\hat{x} = 0$ is a global minimizer.

(3) $f(x) = x$ and so $f'(\hat{x}) = 1$.

Also $h'(x) = \begin{cases} 2x & \text{if } x < 0, \\ 0 & \text{if } x \in [0,1], \\ 2(x-1) & \text{if } x > 1. \end{cases}$

Thus $h'(\hat{x}) = h'(0) = 0$.

Clearly if $\exists \hat{u} \in \mathbb{R}$ such that $\nabla f(\hat{x}) + \hat{u} \nabla h(\hat{x}) = 0$, then we get the contradiction

$$1 = 1 + \hat{u} \cdot 0 = 0.$$

So there cannot exist a $\hat{u} \in \mathbb{R}$ s.t. $\nabla f(\hat{x}) + \hat{u} \nabla h(\hat{x}) = 0$.

But we observe that $\hat{x}$ is not a regular point since $\underbrace{1}_{\neq 0} \cdot \nabla h(\hat{x}) = 1 \cdot 0 = 0$, and so there is no contradiction with Theorem 19.3.

Exercise 19.9.

We want to

$$\text{minimize} \quad 2\pi r^2 + 2\pi r h$$
$$\text{subject to} \quad \pi r^2 h = 1000.$$

We take $h(r,h) := r^2 h - \frac{1000}{\pi}$, and then

we have $\nabla h(r,h) = [2rh \quad r^2] \neq 0$

and so every feasible point is a regular point.

If $(\hat{r}, \hat{h})$ is a minimizer, then $\exists\, \hat{u} \in \mathbb{R}$ s.t.

$$\nabla f(\hat{r}, \hat{h}) + \hat{u}\, \nabla h(\hat{r}, \hat{h}) = 0, \qquad (*)$$

where $f(r,h) := r^2 + rh$.

We have $\nabla f(\hat{r}, \hat{h}) = [2\hat{r} + \hat{h} \quad \hat{r}]$.

Hence $(*)$ becomes

$$\begin{cases} 2\hat{r} + \hat{h} + \hat{u}(2\hat{r}\hat{h}) = 0 \\ \hat{r} + \hat{u}\,\hat{r}^2 = 0. \end{cases}$$

So $\hat{u}\hat{r} = -1$

i.e., $\hat{u} = -\frac{1}{\hat{r}}$.

Hence $2\hat{r} + \hat{h} + \left(-\frac{1}{\hat{r}}\right)(2\hat{r}\hat{h}) = 0$

i.e., $\hat{h} = 2\hat{r}$.

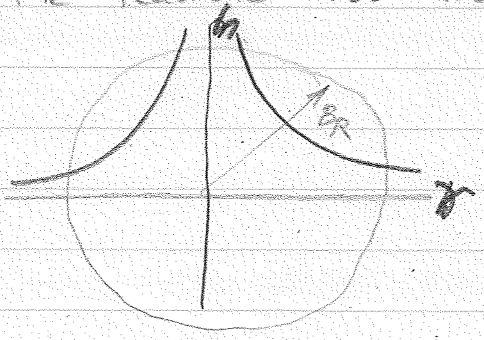
But $\pi \hat{r}^2 \hat{h} = 1000$

So $\pi\, 2\hat{r}^3 = 1000$

i.e., $\hat{r} = \frac{10}{\sqrt[3]{2\pi}}$,

and $\hat{h} = 2\hat{r} = \frac{20}{\sqrt[3]{2\pi}}$.

The feasible set looks like this:



With a large enough R, we can ensure that $f(r,h) > f(\hat{r}, \hat{h})$ for all $(r,h) \notin B_R$ (Why?). B_R is compact and so f assumes a minimum value. It must then be at $(\hat{r}, \hat{h})$.

Exercise 19.10

The problem reduces to the following:

maximize $(\frac{P}{2}-a)(\frac{P}{2}-b)(\frac{P}{2}-c)$

subject to $a+b+c=P$.

We define

$$f(a,b,c) = (\tfrac{P}{2}-a)(\tfrac{P}{2}-b)(\tfrac{P}{2}-c)$$

and $g(a,b,c) = a+b+c-P$.

Then $\nabla g(a,b,c) = [1 \quad 1 \quad 1] \neq 0$,

and so every feasible point is a regular point.

If $(\hat{a},\hat{b},\hat{c})$ is a local maximizer, then $\exists \hat{u} \in \mathbb{R}$ s.t.

$$\nabla f(\hat{a},\hat{b},\hat{c}) + \hat{u}\,\nabla g(\hat{a},\hat{b},\hat{c}) = 0 \qquad (*)$$

We have

$$\nabla f(\hat{a},\hat{b},\hat{c}) = \left[-(\tfrac{P}{2}-\hat{b})(\tfrac{P}{2}-\hat{c}) \quad -(\tfrac{P}{2}-\hat{a})(\tfrac{P}{2}-\hat{c}) \quad -(\tfrac{P}{2}-\hat{a})(\tfrac{P}{2}-\hat{b})\right]$$

So we obtain

$$\hat{u} = (\tfrac{P}{2}-\hat{b})(\tfrac{P}{2}-\hat{c}) = (\tfrac{P}{2}-\hat{a})(\tfrac{P}{2}-\hat{c}) = (\tfrac{P}{2}-\hat{a})(\tfrac{P}{2}-\hat{b})$$

Hence

$$\hat{u}\,(\tfrac{P}{2}-\hat{a}) = (\tfrac{P}{2}-\hat{b})\,\hat{u} = (\tfrac{P}{2}-\hat{c})\,\hat{u}$$

So if $\hat{u} \neq 0$, then $\hat{a} = \hat{b} = \hat{c}$. And using $\hat{a}+\hat{b}+\hat{c} = P$, we then obtain $\hat{a} = \hat{b} = \hat{c} = \frac{P}{3}$.

So the triangle is equilateral in this case.

If $\hat{u} = 0$, then $\nabla f(\hat{a},\hat{b},\hat{c}) = 0$ (from $(*)$) and we have $(\frac{P}{2}-\hat{b})(\frac{P}{2}-\hat{c}) = (\frac{P}{2}-\hat{a})(\frac{P}{2}-\hat{c}) = (\frac{P}{2}-\hat{a})(\frac{P}{2}-\hat{b}) = 0$

Then either $\hat{a} = 0$ or $\hat{b} = 0$ or $\hat{c} = 0$.

and correspondingly the other two sides are equal to $\frac{P}{2}$ each,

So possible local extremizers are:

$$P_0 = (\tfrac{P}{3}, \tfrac{P}{3}, \tfrac{P}{3})$$

$$P_1 = (\tfrac{P}{2}, \tfrac{P}{2}, 0)$$

$$P_2 = (\tfrac{P}{2}, 0, \tfrac{P}{2})$$

$$P_3 = (0, \tfrac{P}{2}, \tfrac{P}{2}).$$

But $f(P_1) = f(P_2) = f(P_3) = 0 < f(P_0)$.

By eliminating c (writing $c = P - a - b$), we have the problem

$$\text{maximize} \quad \left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(a + b - \frac{P}{2}\right).$$

Consider the function

$$(a,b) \overset{F}{\longmapsto} \left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(a + b - \frac{P}{2}\right).$$

If $0 \le a \le \frac{P}{2}$ and $b > \frac{P}{2}$, then $F(a,b) = (+)(-)(+) < 0$.
If $0 \le a \le \frac{P}{2}$ and $b < 0$, then $F(a,b) = (+)(+)(-) < 0$.

So F assumes its maximum on the compact set $\left[0, \frac{P}{2}\right] \times \left[0, \frac{P}{2}\right]$.

Exercise 19.11

Let $h(x,y) := qx + py - b$

Then $\nabla h(x,y) = [q \quad p] \neq 0$.

So every feasible point is regular.

If $(\hat{x},\hat{y})$ is optimal, then $\exists \hat{u} \in \mathbb{R}$ s.t.

$$\left[\frac{\partial Q}{\partial x}(\hat{x},\hat{y}) \quad \frac{\partial Q}{\partial y}(\hat{x},\hat{y})\right] + \hat{u}\,[q \quad p] = 0.$$

Hence $$\frac{1}{q}\frac{\partial Q}{\partial x}(\hat{x},\hat{y}) = -\hat{u} = \frac{1}{p}\frac{\partial Q}{\partial y}(\hat{x},\hat{y}).$$

Exercise 19.12

The problem can be rewritten as follows:

$$\begin{cases} \text{minimize} & f(x) := -x_5 \\ \text{subject to} & h_1(x) := x_1 + x_2 + x_3 + x_4 + x_5 - 8 = 0, \\ & h_2(x) := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 16 = 0, \\ & x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}. \end{cases}$$

We have

$$\nabla h_1(x) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\nabla h_2(x) = \begin{bmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 \end{bmatrix}.$$

Suppose α and β are scalars, not both zeros, such that

$$\alpha \nabla h_1(x) + \beta \nabla h_2(x) = 0.$$

Since $\nabla h_1(x) \neq 0$, it follows that $\beta \neq 0$, and so $\nabla h_2(x) = k \nabla h_1(x)$ for some scalar k. Hence $x_1 = \dots = x_5$. But then $h_1(x) = 0$ gives:

$$x_1 = \dots = x_5 = \frac{8}{5},$$

and then $h_2(x) = 5 \cdot \frac{64}{25} - 16 \neq 0$. So $\nabla h_1(x)$ and $\nabla h_2(x)$ are independent for every feasible x, and so every feasible x is a regular point.

Thus if x is a local optimal solution, then there exists a

$$u = \begin{bmatrix} \lambda \\ \mu \end{bmatrix} \in \mathbb{R}^2$$

such that $\nabla f(x) + u^T \nabla h(x) = 0$, that is,

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} \lambda & \mu \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2x_1 & 2x_2 & 2x_3 & 2x_4 & 2x_5 \end{bmatrix} = 0.$$

Hence

$$\lambda + 2\mu x_1 = 0 \qquad (1)$$

$$\lambda + 2\mu x_2 = 0 \qquad (2)$$

$$\lambda + 2\mu x_3 = 0 \qquad (3)$$

$$\lambda + 2\mu x_4 = 0 \qquad (4)$$

$$-1 + \lambda + 2\mu x_5 = 0. \qquad (5)$$

We consider the two cases $\lambda = 0$ and $\lambda \neq 0$ separately.

1° If $\lambda = 0$, then (5) gives $2\mu x_5 = 1$ and so $\mu \neq 0$. But then (1)-(4) give $x_1 = x_2 = x_3 = x_4 = 0$. So $h_1(x) = 0$ now gives $x_5 = 8$. But then $h_2(x) = 64 - 16 \neq 0$. So this case gives no feasible x.

2° Suppose $\lambda \neq 0$. Then (1) gives $2\mu x_1 = -\lambda$ and so $\mu \neq 0$. Then (1)-(4) give $x_1 = x_2 = x_3 = x_4 = \frac{-\lambda}{2\mu} = k$ (say). Then $h_1(x) = 0$ gives $4k + x_5 - 8 = 0$ while $h_2(x) = 0$ gives $4k^2 + x_5^2 - 16 = 0$. Eliminating k, we obtain

$$x_5^2 + 4\left(\frac{8 - x_5}{4}\right)^2 - 16 = 0,$$

and upon simplifying, we obtain $x_5\left(\frac{5}{4}x_5 - 4\right) = 0$. Thus $x_5 = \frac{16}{5}$ or $x_5 = 0$. Hence $x = \left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5}\right)$ or $x = (2, 2, 2, 2, 0)$. Both of these are feasible, and since $\frac{16}{5} > 0$, we conclude that if there is an optimal solution, it must be $x = \left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5}\right)$.

The feasible set $\mathcal{F}$, namely
$\{x \in \mathbb{R}^5 : x_1 + x_2 + x_3 + x_4 + x_5 = 8\} \cap \{x \in \mathbb{R}^5 : x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 16\}$
is bounded (indeed, $\mathcal{F}$ is contained in the ball with center 0 and radius 4), and it is also closed (since it is the intersection of two closed sets).
So $\mathcal{F}$ is compact. The map $x \mapsto -x_5$ is continuous. So we know that $f : \mathcal{F} \to \mathbb{R}$ has a global minimum

on $\mathcal{J}_e$. Consequently, $x = \left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{6}{5}, \frac{16}{5}\right)$ is a global minimizer.

So the largest value of x_5 is $\frac{16}{5}$.