Exercise 24.4

(1) $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 3 & 1 & 2 \end{bmatrix} \xrightarrow{r_2 - 3r_1} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & -1/2 & 7/2 \end{bmatrix}$

$\xrightarrow{-2r_2} \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 1 & -7 \end{bmatrix}$

$\xrightarrow{r_1 - 1/2 r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix}$

Thus we define $C := U := \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix}$, and

$$B := A_\beta := \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}.$$

(Then $BC = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix} = A$.)

(2) $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}r_2} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -4 \end{bmatrix} \xrightarrow[r_4 + 4r_2]{r_1 - r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus we define $C := U := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and

$$B := A_\beta := \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}.$$

(Then $BC = A$.)

Exercise 24.5

(See the solution to Exercise 25.4.)

(1) A basis for

$\operatorname{ran} \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$ is given by $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

A basis for

$\operatorname{ran} \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -7 \end{bmatrix} \right\}$.

(2) A basis for

$\operatorname{ran} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$ is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

A basis for

$\operatorname{ran} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Exercise 24.6

(See the solution to Exercise 25.4.)

(1) $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$. Here $k = n - r = 3 - 2 = 1$.

$\ker A = \{ x \in \mathbb{R}^3 : x_\beta = -U_\nu x_\nu \}$.

$$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}.$$

Hence $\ker A = \operatorname{span} \begin{bmatrix} -3 \\ 7 \\ 1 \end{bmatrix}$.

(2) $$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U}.$$

Here $k = n - r = 2 - 2 = 0$.

So $\ker A = \{0\}$.

Exercise 24.7

(1) $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{bmatrix}$;

$$A^T = \begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2} r_1} \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \xrightarrow[r_3 + r_1]{r_2 - r_1} \begin{bmatrix} 1 & 3/2 \\ 0 & -1/2 \\ 0 & 7/2 \end{bmatrix} \xrightarrow{-2 r_2} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \\ 0 & 7/2 \end{bmatrix}$$

$$\xrightarrow[r_3 - \frac{7}{2} r_2]{r_1 - \frac{3}{2} r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

So $A^T = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \\ -1 & 2 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{U}$.

$\ker A^T = \ker U = 0$.

(2) $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{bmatrix}$; $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2} r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{r_1 - 2 r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

So $A^T = \underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}}_{U}$,

We have $\dim(\ker A^T) = n - r = 3 - 2 = 1$.

$\ker A^T = \{ x \in \mathbb{R}^3 : x_\beta = -U_\nu x_\nu \}$,

$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot 1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Hence $\ker A^T = \operatorname{span} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Exercise 24.9

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & 1 \end{bmatrix}$$

$$\xrightarrow[r_3 - 2r_2]{r_1 - r_2} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 2 & 4 \end{bmatrix}}_{A_\beta} \underbrace{\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}}_{U}$.

So a basis for ran A is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \right\}$, and

a basis for ran A^T is $\left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \right\}$.

We have $\dim(\ker A) = n - r = 3 - 2 = 1$.

$\ker A = \{ x \in \mathbb{R}^3 : x_\beta = -U_\nu x_\nu \}$.

$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot 1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Hence $\ker A = \operatorname{span} \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$.

$$A^T = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2} r_2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[r_3 - r_2]{r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $A^T = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Again $\dim(\ker A^T) = n - r = 3 - 2 = 1$.

$\ker A^T = \{ x \in \mathbb{R}^3 : x_\beta = -U_\nu x_\nu \}$.

$x_\beta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Hence $\ker A^T = \operatorname{span} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.