Exercise 25.5

$$x^T H x = [x_1 \quad x_2]\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \quad x_2]\begin{bmatrix} x_1 - x_2 \\ -x_1 + 4x_2 \end{bmatrix}$$

$$= x_1^2 - x_1x_2 - x_1x_2 + 4x_2^2$$

$$= x_1^2 - 2x_1x_2 + x_2^2 + 3x_2^2$$

$$= (x_1 - x_2)^2 + 3x_2^2$$

$$\geqslant 0$$

and if $x^T H x = 0$, then $(x_1 - x_2)^2 + 3x_2^2 = 0$, and so

$x_1 = x_2$ and $x_2 = 0$ i.e., $x = 0$.

Hence H is positive definite.

Exercise 25.11

(1) If $1 \leq k \leq n$, then take $x \in \mathbb{R}^n$ of the form

$x = \begin{bmatrix} \xi \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, where $\xi \in \mathbb{R}^k$. Denote the $k \times k$ principal submatrix of H by H_k.

Then $0 \leq x^T H x = [\xi^T \; 0^T] \begin{bmatrix} H_k & * \\ * & * \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = [\xi^T \; 0^T] \begin{bmatrix} H_k \xi \\ * \end{bmatrix} = \xi^T H_k \xi$.

Hence $\xi^T H_k \xi \geq 0 \quad \forall \xi \in \mathbb{R}^k$. Hence H_k is positive semidefinite. Also if $\xi^T H_k \xi = 0$, then $x^T H x = 0$, where $x := \begin{bmatrix} \xi \\ 0 \end{bmatrix}$. By the positive definiteness of H, $x = 0$. Hence $\xi = 0$. So H_k is positive definite. By Property 26.11, it follows that all eigenvalues of of H_k are positive, and so $\det H_k$ is positive too.
(Here we use the fact that for a diagonalizable matrix A, $\det A$ = product of its eigenvalues.
Indeed one way to see this is as follows: if $A = PDP^{-1}$, then $\det A = \det(PDP^{-1}) = (\det P)(\det D)(\det P^{-1}) = (\det D)(\det(PP^{-1}))$
$= \det D$.)

(2) Let $S_1 := W$ and $S_2 := \text{span}\{v_{m+1}, \dots, v_n\}$.
Then by the result of Exercise 23.9, we have

$$\begin{aligned} \dim(S_1 \cap S_2) &= -\dim(S_1 + S_2) + \dim S_1 + \dim S_2 \\ &\geq -n + k + n - m \\ &= k - m > 0. \end{aligned}$$

Hence S_1 has a nontrivial intersection with S_2, that is, there is a nonzero vector in W which is a linear combination of $v_{m+1}, \dots, v_n$.

(3) Suppose that $\omega^T H \omega > 0$ for all nonzero vectors ω in a k-dimensional subspace W of $\mathbb{R}^n$. By the spectral theorem, H has an orthonormal basis of eigenvectors $v_1, \dots, v_n$. Suppose that the first m of these eigenvectors are the ones corresponding to positive eigenvalues, while the others correspond to nonpositive eigenvalues $\lambda_{m+1}, \dots, \lambda_n$. Suppose that $m < k$. Then by the preview part of this exercise, there is a nonzero ω which is a linear combination of $v_{m+1}, \dots, v_n$, say

$$\omega = \alpha_{m+1} v_{m+1} + \dots + \alpha_n v_n .$$

Then
$$\omega^T H \omega = (\alpha_{m+1} v_{m+1}^T + \dots + \alpha_n v_n^T)(\alpha_{m+1} \lambda_{m+1} v_{m+1} + \dots + \alpha_n \lambda_n v_n)$$
$$= \alpha_{m+1}^2 \lambda_{m+1} + \dots + \alpha_n^2 \lambda_n$$
$$\leq 0,$$
a contradiction. Hence $m \geq k$.

(4). If H is 1×1, then this is obvious: since $x^T H x = x^2 (\det H) > 0$ for all nonzero $x \in \mathbb{R}^1$.

Suppose the result is true for $n \times n$ matrices. Now let $H \in \mathbb{R}^{(n+1) \times (n+1)}$ be such that all principal minors of H are positive definite. In particular, all n principal minors of H_n are positive definite, and so H_n is positive definite by the induction hypothesis. (Here H_n denotes the nth principal submatrix of H.) Let $W = \operatorname{span}\{e_1, \dots, e_n\}$. Then if $\omega \in W$, we have $\omega = \begin{bmatrix} \xi \\ 0 \end{bmatrix}$, where $\xi \in \mathbb{R}^n$. Hence $\omega^T H \omega = \xi^T H_k \xi > 0$ if $\xi \neq 0$. Hence $\omega^T H \omega > 0$ for all nonzero vectors ω in W. By the previous part of this exercise, it follows that at least n eigenvalues of H must be positive. But the product of all $n+1$ eigenvalues of H is equal to $\det H > 0$. Hence all eigenvalues of H are positive,

Exercise 25.13

$\det A_1 = \det [4] = 4 > 0,$

$$\det A_2 = \det \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} = 12 - 4 = 8 > 0,$$

$$\det A_3 = \det A = \det \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix} = 4(6-1) - 2(4+1) + 1(-2-3)$$

$$= 20 - 10 - 5 = 5 > 0.$$

Hence A is positive definite.

$\det B_1 = \det [3] = 3 > 0,$

$$\det B_2 = \det \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} = 12 - 1 = 11 > 0,$$

but
$$\det B_3 = \det B = \det \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & -2 \\ 2 & -2 & 1 \end{bmatrix} = 3(4-4) + 1(-1+4) + 2(2-8)$$

$$= 3 + (-12) = -9 < 0.$$

So B is not positive definite.

Clearly $-A$ is not positive definite, since for nonzero x, $x^T(-A)x = -x^TAx < 0$.
Since A is positive definite, its eigenvalues are all positive. But the eigenvalues of A^3, being the cubes of these, are also positive. So A^3 is positive definite.
Finally the eigenvalues of A^{-1} are reciprocals of the eigenvalues of A, and so A^{-1} is positive definite as well. (Or directly: $\forall y, \exists! x$ s.t. $Ax = y$. So $y^TA^{-1}y = x^TAA^{-1}Ax = x^TAx > 0$ for all nonzero y.)

Exercise 25.14

(1) TRUE. (If $A = P^T D P$, then $A^5 = P^T D^5 P$. D has positive entries, and so does D^5.)

(2) FALSE (Take for example, $A = -I$. Then $A^8 = I$.)

(3) TRUE. (If $A = P^T D P$, then D has negative entries. Then $A^{12} = P D^{12} P$, and D^{12} has positive entries.)

(4) TRUE. (Let $x \neq 0$. Then $x^T A x > 0$, $x^T B x \leq 0$. Hence $x^T (A - B) x = x^T A x - x^T B x > 0$.)

Exercise 25.23

$$H = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 5 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}.$$

Let $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then $E_1 H E_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$.

Let $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$. Then $E_2 E_1 H E_1^T E_2^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

So H is neither.

Let $E_1' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then $E_1' H' E_1'^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3 \end{bmatrix}$.

Let $E_2' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$. Then $E_2' E_1' H' E_1'^T E_2'^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

So H' is positive semi definite.