TAKE HOME ASSIGNMENT II

1. Instructions

In this home assignment, it is allowed to work in groups of at most three members. One report per group is to be handed in, in which you should solve the exercises given below. A print out of the MATLAB file and plots should be included in the report.

State the name, personal number and e-mail address of all members of the group on the front page of the report.

The report should be handed in to me in the Lecture (08:00-10:00) on Tuesday, 5 October, 2010. The deadline is sharp.

2. Introduction

As in Take home assignment I, we will consider the problem of spectral estimation.

2.1. The periodogram estimate. Suppose that a data sequence

$$y_0, \dots, y_N$$

is given. In MATLAB, an estimate of the *spectral density* Φ of the signal $y = (y_0, \dots, y_N)$ is given by

```
>> Phi = abs(fft(y)).^2;
```

(Roughly speaking, the spectral density gives the energy content of the signal y at the frequency θ.) This estimate is referred to as the *periodogram estimate* of the sound signal y. It is common to use deciBels (dB) to express the spectral density of sound signals. So one defines

$$\Psi = 10 \log_{10} \Phi.$$

2.2. A rational model. Now the task is to fit a rational model to the estimated spectral density Ψ above, that is we want to find a $\widehat{\Psi}$, such that $\widehat{\Psi}$ is close to Ψ, and moreover,

$$\widehat{\Psi}(\theta) = \frac{S(\theta)}{T(\theta)}, \quad \theta \in [0, 2\pi],$$

where

$$\begin{aligned} S(\theta) &= p_0 + p_1 \cos\theta + \cdots + p_n \cos(n\theta), \\ T(\theta) &= 1 + q_1 \cos\theta + \cdots + q_n \cos(n\theta). \end{aligned}$$

Why would one want to do this? The aim is that the long data y (which has N components), is now replaced by the short data of coefficients $p_0, \dots, p_n$ and $q_1, \dots, q_n$, and if $2n + 1 \ll N$, we have saved a lot of memory space. This idea is used in speech processing for characterizing short speech segments. The point is that if we compare two samples of a person uttering the same word twice, then the data in the time domain is quite different. But the frequency domain picture is very similar, and an approximation of the frequency domain picture is enough for reproduction of that sound. Thus it makes sense to approximate the Ψ. In this manner, the rational model can be used for speech recognition, speaker recognition, speech compression and so on.

2.3. **The problem.** Just as in Take-home Assignment I, one would like to minimize the norm of $\|T(\Psi - \widehat{\Psi})\| = \|T\Psi - S\|$. But the difference with the Take-home Assignment I is that now we will use the 2-norm. And we want to use quadratic programming to solve the problem.

To begin with, for our y, we will use a standard sound signal in Matlab. This sound signal is called *handel*, which is a sample from a choir singing Händel's Messiah. In order to get the periodogram estimate Ψ for this y, we proceed as follows:

```
>> load handel;
>> N = 1024;
>> Psi = 10*log10((1/N)*abs(fft(y(10001:10000+N))).^2);
>> Psi = Psi(1:N/2);
```

(One can listen to the y we are using using the command `sound(y,Fs)`. Since our signal y is a real sequence, it is enough to take the first $\frac{N}{2}$ components of Ψ; this is the reason for the $N/2$ in the fourth MATLAB command above.)

This gives us a vector with values $\Psi(\theta_k)$, where

$$\theta_k = \frac{k\pi}{\frac{N}{2}}, \quad k = 0, \dots, \frac{N}{2} - 1.$$

We will use the ℓ^2-norm for sequences. Thus if we have a sequence $f = (f_k)_{0 \le k \le \frac{N}{2}-1}$, then

$$\|f\|_2 := \sqrt{\sum_{k=0}^{\frac{N}{2}-1} |f_k|^2}.$$

Since we are considering the problem of minimization of the norm of $\Psi T - S$, we are led to the following optimization problem:

$$(Q): \text{ minimize } \left\| \left(\Psi(\theta_k) \cdot T(\theta_k) - S(\theta_k)\right)_{0 \le k \le \frac{N}{2}-1} \right\|_2. \tag{1}$$

The variables in the above minimization problem are the coefficients $p_0, \dots, p_n$ and $q_1, \dots, q_n$.

3. Exercises

(1) Recast the optimization problem (Q) in (1) as a quadratic programming problem.

Motivate your formulation, that is, explain why your quadratic programming problem enables one to solve the original optimization problem (Q). In your quadratic programming problem, you should specify clearly the variables, the constraints and the objective function.

(2) Write a program in MATLAB to solve your quadratic programming problem. Use $n = 8$.

(3) Plot $(\Psi(\theta_k) \cdot T(\theta_k))_{0 \le k \le \frac{N}{2}-1}$ and $(S(\theta_k))_{0 \le k \le \frac{N}{2}-1}$ in the same graph. What do you observe?

(4) Now we will add a constraint in our problem. Suppose that we want a solution where

$$\sum_{k=0}^{n} p_k = 0 \text{ and } \sum_{k=1}^{n} q_k = 0.$$

(a) Recast the new optimization problem as a quadratic programming problem with constraints.
(b) Write a program in MATLAB to solve your quadratic programming problem using the Lagrange method.
(c) Plot $(\Psi(\theta_k) \cdot T(\theta_k))_{0 \le k \le \frac{N}{2}-1}$ and $(S(\theta_k))_{0 \le k \le \frac{N}{2}-1}$ in the same graph.