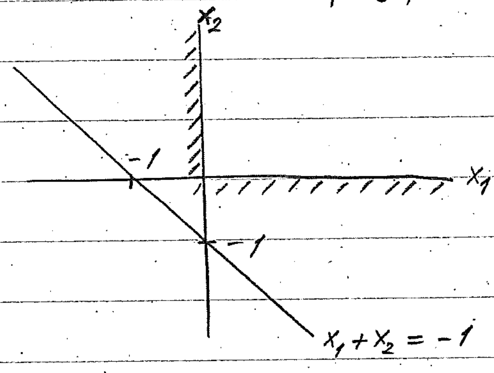
Exercise 4.2.

Not every linear programming problem in standard form has a nonempty feasible set. For example, consider:

$$\begin{cases} \text{minimize} & *** \\ \text{subject to} & x_1 + x_2 = -1 \\ & x_1 \geq 0 \text{ and } x_2 \geq 0. \end{cases}$$

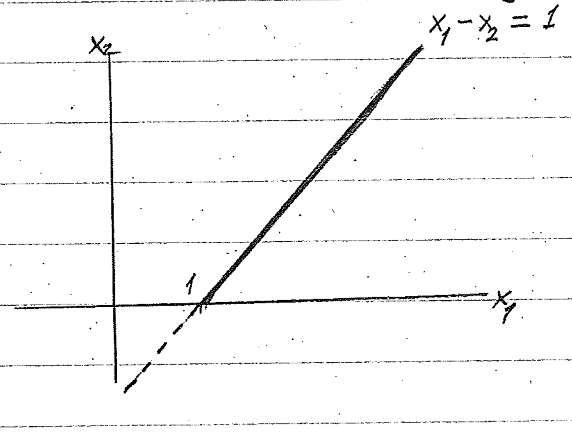
The feasible set is empty, and this is visibly clear:



Not every linear programming problem in the standard form has an optimal solution. For example, consider:

$$\begin{cases} \text{minimize} & -x_2 \\ \text{subject to} & x_1 - x_2 = 1 \\ & x_1 \geq 0 \text{ and } x_2 \geq 0 \end{cases}$$

Then the point $(1+t, t)$ is feasible for $t \geq 0$. But $-x_2 = -t \searrow -\infty$ as $t \nearrow +\infty$. So the optimal value of the problem is $-\infty$, and clearly then there is no optimal solution. See the figure below:



Exercise 4.7

We have rank $A = 2$ (for example $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are linearly independent). In principle, there can be $\binom{5}{2} = \frac{5 \times 4}{2 \times 1} = 10$ basic feasible solutions.

β	A_β	rank A_β	x_β $(A_\beta x_\beta = b)$	Remarks
(1, 2)	$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$	2	$\begin{bmatrix} 33/5 \\ -4/5 \end{bmatrix}$	Basic, but not feasible
(1, 3)	$\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$	2	$\begin{bmatrix} 8 \\ -1 \end{bmatrix}$	Basic, but not feasib
(1, 4)	$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$	2	$\begin{bmatrix} 19/3 \\ -4/3 \end{bmatrix}$	Basic, but not feasibl
(1, 5)	$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$	2	$\begin{bmatrix} 5 \\ 4/3 \end{bmatrix}$	Basic feasible solutio
(2, 3)	$\begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$	2	$\begin{bmatrix} -32/7 \\ 33/7 \end{bmatrix}$	Basic, but not feasibl
(2, 4)	$\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$	2	$\begin{bmatrix} 19 \\ -33 \end{bmatrix}$	Basic, but not feasib
(2, 5)	$\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$	2	$\begin{bmatrix} 5/2 \\ 11/2 \end{bmatrix}$	Basic feasible solutio
(3, 4)	$\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$	2	$\begin{bmatrix} 19/5 \\ -32/5 \end{bmatrix}$	Basic, but not feasibl
(3, 5)	$\begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$	2	$\begin{bmatrix} 5/3 \\ 32/9 \end{bmatrix}$	Basic feasible solution
(4, 5)	$\begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$	2	$\begin{bmatrix} 5 \\ 19/3 \end{bmatrix}$	Basic feasible solution

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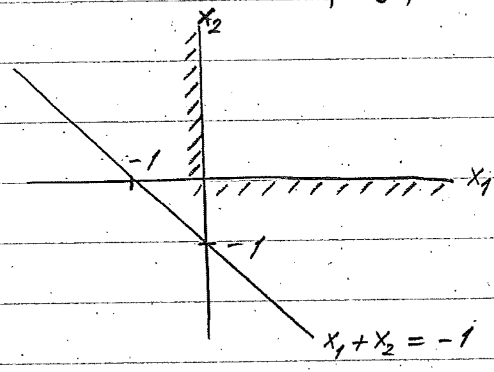

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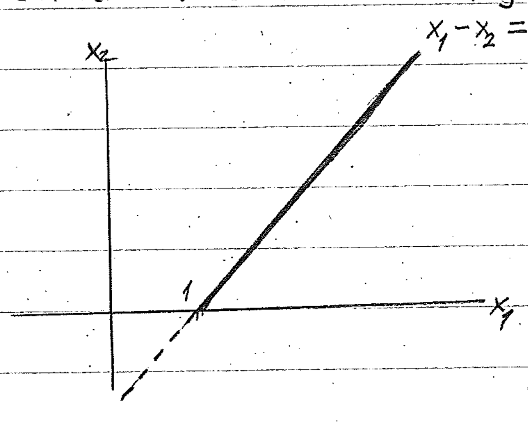

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Exercise 4.9

We consider the two possible cases:

1° $\exists\, v \in \mathbb{R}^m$ s.t. $c^T = v^T A$.

Then $c^T x_0 = v^T A x_0 = v^T b$. $(*)$

Since (P) has a feasible solution, it has a basic feasible solution, say x_*. Then x_* has at most m positive components, and

$$c^T x_* = v^T A x_* = v^T b \underset{\substack{\uparrow \\ \text{from } (*)}}{=} c^T x_0 .$$

2° $\neg\left[\exists\, v \in \mathbb{R}^m \text{ s.t. } c^T = v^T A\right]$. Let $A' = \begin{bmatrix} A \\ c^T \end{bmatrix}$.

Then rank $A' = m+1$. For if not, $\exists\, v \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, not both zero such that $v^T A + \alpha c^T = 0$.

Now if $\alpha = 0$, then $v^T A = 0$ and so $v = 0$ since rank $A = m$. So $\alpha \neq 0$. But then $c^T = -\frac{1}{\alpha} v^T A$, a contradiction to our hypothesis.

Now consider the auxiliary problem:

$$(P') \begin{cases} \text{minimize} & c^T x \\ \text{subject to} & A' x = \begin{bmatrix} A \\ c^T \end{bmatrix} x = b' := \begin{bmatrix} b \\ c^T x_0 \end{bmatrix} \\ & x \geqslant 0 . \end{cases}$$

(P′) has a feasible solution, namely x_0.

So it has a basic feasible solution x_{**} with at most $m+1$ components. But then $A' x_{**} = \begin{bmatrix} A \\ c^T \end{bmatrix} x_{**} = \begin{bmatrix} b \\ c^T x_0 \end{bmatrix}$;

and in particular, $A x_{**} = b$, and $c^T x_{**} = c^T x_0$.

Also $x_{**} \geqslant 0$.

So x_{**} is a feasible solution to (P) and $c^T x_{**} = c^T x_0$.

Exercise 4.12

Let $x, y \in C$ and $t \in (0,1)$. Let $i \in I$.
Then $x, y \in \left(\bigcap_{i \in I} C_i \subset \right) C_i$. Since C_i is convex, $(1-t)x + ty \in C_i$. But the choice of $i \in I$ was arbitrary. Hence $(1-t)x + ty \in \bigcap_{i \in I} C_i = C$.
So C is convex.

Exercise 4.13.

If $n=1$ and $x_1 \in C$, then $\frac{x_1}{1} = x_1 \in C$.
Suppose the result is true for some n.
Thus whenever $x_1, \dots, x_n \in C$, also $\frac{x_1 + \dots + x_n}{n} \in C$.

Suppose we are given $x_1, \dots, x_n, x_{n+1} \in C$.
Then $x := \frac{x_1 + \dots + x_n}{n} \in C$ and $y := x_{n+1} \in C$.

Take $t = \frac{1}{n+1}$. Then

$(1-t)x + ty \in C$, i.e., $\left(1 - \frac{1}{n+1}\right) \cdot \frac{x_1 + \dots + x_n}{n} + \frac{1}{n+1} x_{n+1} \in C$

i.e., $\frac{n}{n+1} \cdot \frac{x_1 + \dots + x_n}{n} + \frac{1}{n+1} x_{n+1} \in C$

i.e., $\frac{x_1 + \dots + x_n + x_{n+1}}{n+1} \in C$.

By induction, the result follows.

Exercise 4.23

Let E, E' be the set of extreme points of C, C', respectively. If $e \in E$, then we first show that $Te \in E'$. Suppose that $Te = (1-t)a' + tb'$ for some $a', b' \in C'$ such that $a' \neq b'$ and $t \in (0,1)$. Then there are $a, b \in C$ such that $Ta = a'$ and $Tb = b'$. Clearly $a \neq b$, for otherwise $a' = b'$. Now we have $Te = (1-t)Ta + tTb = T((1-t)a + tb)$. Since $e, (1-t)a + tb \in C$ and $T: C \to C'$ is one-to-one, it follows that $e = (1-t)a + tb$. But since $a \neq b$ and $t \in (0,1)$ this is a contradiction to the fact that $e \in E$. Hence $Te \in E'$. So the map $T: E \to E'$ is well-defined.

Next we show that if $e' \in E'$, then there exists a unique $e \in E$ such that $Te = e'$. First of all $e' \in E' C'$ and so we know that there exists a unique e at least in C such that $Te = e'$. We claim that in fact $e \in C$. Suppose, on the contrary, that there are $a, b \in C$ such that $a \neq b$ and there exists a $t \in (0,1)$ such that $e = (1-t)a + tb$. Then $Te = T((1-t)a + tb) = (1-t)Ta + tTb$. Also, since $a \neq b$, $Ta \neq Tb$ (as T is one-to-one). Moreover $Ta, Tb \in C'$. This means that $e' = Te$ is not an extreme point of C', a contradiction. Thus $e \in E$.

Consequently, $T: E \to E'$ establishes a one-to-one correspondence between the extreme points of C and C'.

Exercise 4.24

Consider the two convex sets

$\mathcal{F} := \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ $(\subset \mathbb{R}^n)$, and

$\mathcal{F}' := \{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{n+m} : [A \ \ I] \begin{bmatrix} x \\ y \end{bmatrix} = b,\ x \geq 0, y \geq 0\}$ $(\subset \mathbb{R}^{n+m})$

Consider the map $T: \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ given by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = x \left(= [I \ \ 0] \begin{bmatrix} x \\ y \end{bmatrix} \right).$$

(That is, T is the linear transformation given by matrix multiplication by $[I \ \ 0] \in \mathbb{R}^{n \times (n+m)}$.)

We claim that T establishes a one-to-one correspondence between $\mathcal{F}'$ and $\mathcal{F}$.

Let $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{F}'$, i.e., $Ax + y = b$, $x \geq 0$ and $y \geq 0$. We have $T \begin{bmatrix} x \\ y \end{bmatrix} = [I \ \ 0] \begin{bmatrix} x \\ y \end{bmatrix} = x$. Now $Ax = b - y \leq b$ (since $y \geq 0$) and $x \geq 0$. Thus $x \in \mathcal{F}$ i.e, $T \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{F}$.

Hence $T: \mathcal{F}' \longrightarrow \mathcal{F}$ is well-defined.

Next we show that for every $x \in \mathcal{F}$, there is a unique $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathcal{F}'$ such that $T \begin{bmatrix} x' \\ y' \end{bmatrix} = x$.

First we show uniqueness. Suppose there are $\begin{bmatrix} x' \\ y' \end{bmatrix}, \begin{bmatrix} x'' \\ y'' \end{bmatrix}$ in $\mathcal{F}'$ such that $T \begin{bmatrix} x' \\ y' \end{bmatrix} = T \begin{bmatrix} x'' \\ y'' \end{bmatrix} = x$.

Then $x' = x'' = x$. Also $Ax' + y' = b = Ax'' + y''$. Since $x' = x'' = x$, we have $y' = y'' = b - Ax$. Hence $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \end{bmatrix}$.

To show the claimed existence, just take $x' := x$ and $y' := b - Ax$. Then $x' = x \geq 0$ and $y' = b - Ax \geq 0$ (since $Ax \leq 0$). Moreover, $[A \ \ I] \begin{bmatrix} x' \\ y' \end{bmatrix} = Ax' + y' = Ax + b - Ax = b$

Thus $\begin{bmatrix} x' \\ y' \end{bmatrix} \in \mathcal{F}$.

Consequently, $T: \mathcal{F}' \longrightarrow \mathcal{F}$ gives a one-to-one correspondence between $\mathcal{F}'$ and $\mathcal{F}$.

By the previous exercise, T also gives a one-to-one correspondence between the extreme points of the feasible sets of the two problems.