Optimization

$$f : \mathcal{X} \longrightarrow \mathbb{R}$$

$$\left\{ \begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t} & x \in \mathcal{X} \end{array} \right. \qquad \cancel{\left\{ \begin{array}{ll} \text{maximize} & f(x) \\ \text{s.t} & x \in \mathcal{X} \end{array} \right.}$$

In this course : $\mathcal{X} \subset \mathbb{R}^n$

Part I

Linear programming

$$f(x) = c^T x$$

$$\mathcal{X} = \{ x \in \mathbb{R}^n : Ax \geqslant b \}$$

Part II

Quadratic optimization

$$f(x) = \frac{1}{2} x^T H x + c^T x + c_0$$

$$\mathcal{X} = \{ x : Ax = b \}$$

+

Some linear algebra.

Part III

Nonlinear optimization.

$$f(x)$$

$$\mathcal{X} = \left\{ x : \begin{array}{c} g_1(x) \leq 0 \\ \vdots \\ g_m(x) \leq 0 \end{array} \right\}.$$

(I.) Linear programming

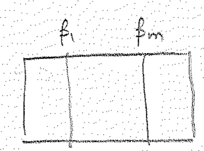
$$\begin{cases} \text{minimize} & c^T x \\ \text{s.t} & Ax \geqslant b \end{cases} \quad \Longrightarrow \quad \underline{\text{Standard form}} \quad \begin{cases} \text{minimize} & c^T x \\ \text{s.t} & Ax = b \\ & x \geqslant 0 . \end{cases}$$

(1) inequalities $\rightsquigarrow$ equalities by introducing slack variables

(2) Replace each free variable by a difference of two new nonnegative variables.

So it is enough to learn to solve problems in the standard form.

$$A x = b$$



$$x = b$$

$$A_\beta x_\beta + A_\nu x_\nu = b$$

A special solution: Set $x_\nu = 0$, and the solve for x_β: $x_\beta = A_\beta^{-1} b$.

This is a <u>basic solution</u>. If a basic solution is $\geqslant 0$, it is called a <u>basic feasible solution</u>.

Fundamental theorem of linear programming

Consider (P): $\begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases}$

If (P) has an optimal solution, then there is a basic feasible soln. which is optimal for (P)

So it is enough to search among the basic feasible solutions.

There are only finitely many basic feasible solutions.

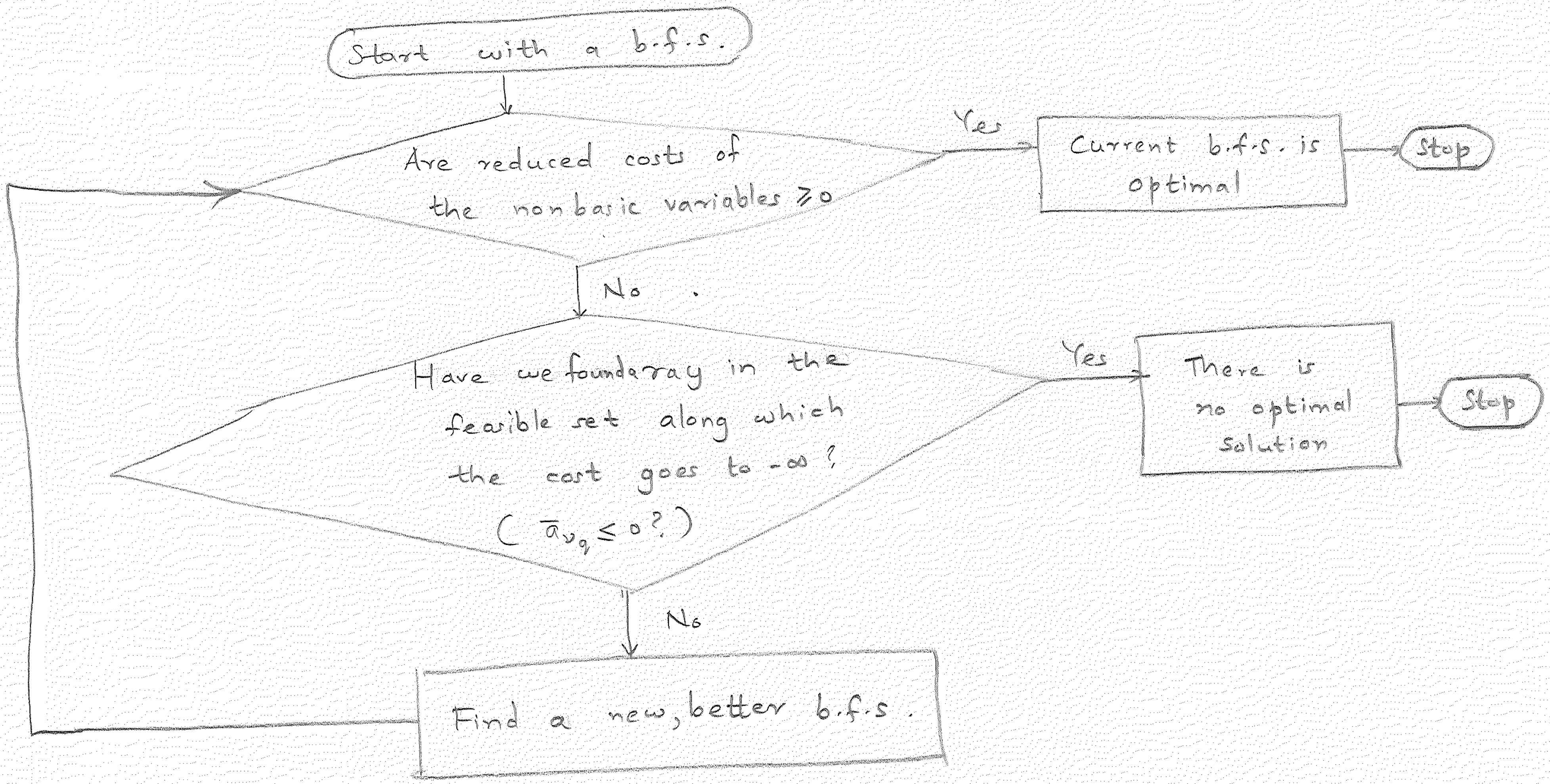
$$\# \text{ basic feasible solutions} \leq \binom{n}{m}.$$

$\binom{n}{m}$ can be very large even for modest n, m

$n = 50$
$m = 5$ $\qquad \binom{n}{m} > 2 \times 10^6$

Simplex method: - a reasonable method for searching among the basic feasible solutions

- does not require calculation of <u>all</u> basic feasible solutions.
- goes from one b.f.s. to a better one in each iteration.


Start with a b.f.s.
Are reduced costs of the nonbasic variables ⩾ 0
Yes
Current b.f.s. is optimal
Stop
No
Have we found a ray in the feasible set along which the cost goes to -∞ ? ($\bar{a}_{v_q} \leq 0$?)
Yes
There is no optimal solution
Stop
No
Find a new, better b.f.s.

Duality theory

$$(P): \begin{cases} \text{min.} & c^T x \\ \text{s.t.} & Ax \geqslant b \\ & x \geqslant 0 \end{cases} \qquad (D): \begin{cases} \text{max.} & b^T y \\ \text{s.t} & A^T y \leqslant c \\ & y \geqslant 0 \end{cases}$$

Primal problem $\rightsquigarrow$ Dual problem

Weak duality: $\left.\begin{array}{l} x \text{ feasible for } (P) \\ y \text{ feasible for } (D) \end{array}\right\} \Rightarrow b^T y \leqslant c^T x$.

Strong duality:

$\mathcal{F}_P = \left\{ x : \begin{array}{l} Ax \geqslant b \\ x \geqslant 0 \end{array} \right\}$	$\mathcal{F}_D = \left\{ y : \begin{array}{l} A^T y \leqslant c \\ y \geqslant 0 \end{array} \right\}$	Conclusion
$\neq \phi$	$\neq \phi$	$\exists$ an opt. soln $\hat{x}$ for (P) $\exists$ an opt. soln $\hat{y}$ for (D) $c^T\hat{x} = b^T\hat{y}$
$\neq \phi$	$\neq \phi$	Neither (P) nor (D) has an optimal soln.
$= \phi$	$\neq \phi$	
$= \phi$	$= \phi$	

<u>Corollary</u>

$$\boxed{\begin{array}{l} \hat{x} \text{ feasible for (P)} \\ \hat{y} \text{ feasible for (D)} \\ c^T\hat{x} = b^T\hat{y} \end{array}} \Longleftrightarrow \boxed{\begin{array}{l} \hat{x} \text{ optimal for (P)} \\ \hat{y} \text{ optimal for (D)} \end{array}}$$

Complimentarity theorem

Dual to LP problem in general form

Dual to LP problem in standard form

$$(P): \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases} \qquad (D): \begin{cases} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leq c \end{cases}$$

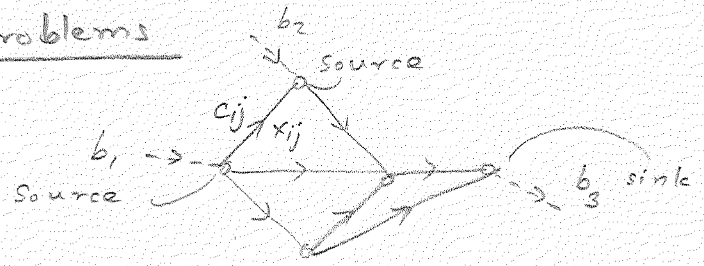
Suppose simplex method used on (P) terminates since $r_\nu \geq 0$.

Know: The last b.f.s. is optimal for (P).

Fact: The last simplex multipliers vector y $(A_\beta^T y = c_\beta)$

is optimal for (D).

Network flow problems



Balanced network: $\sum b_i = 0$

$\sum_{(i,j)\in E} c_{ij}x_{ij}$

$$\begin{cases} \text{minimize} & \text{cost of flow} \\ \text{s.t.} & \text{flow balance at each node} \\ & x \geq 0 \end{cases}$$

$$\begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & \boxed{\tilde{A}x = \tilde{b}} \\ & x \geq 0 \end{cases}$$

$\tilde{A}$ incidence matrix

Does not have independent rows

Delete last row in $\tilde{A}$ to get A

" entry in $\tilde{b}$ " b

Then we obtain problem in standard form

$$\begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{cases}$$

Applying simplex method:

(1) Finding an initial basic solution.

Fact:

$$\begin{tikzcd} \text{spanning tree} \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & m-1 \text{ independent columns of } A \arrow[d, leftrightarrow] \\ & \text{basic solution.} \end{tikzcd}$$

So just choose a spanning tree.

Use flow balance to find a basic solution.

Check if it feasible.

(2) Simplex multiplier vector y.

$$\begin{cases} y_i - y_j = c_{ij} & \text{for tree edges } (i,j) \\ y_m = 0 \end{cases}$$

(3) Reduced costs of nonbasic variables. $r_{ij} = c_{ij} - (y_i - y_j)$ for nontree edges (i,j).

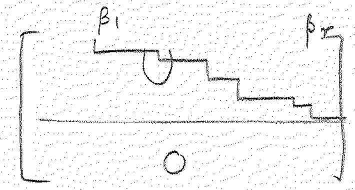
(4) New basic feasible solution. $x_{ij} = t$ for edge (i,j) s.t. r_{ij} is the most -ve component of r_{ij}.

Increase t from 0. Use flow balance.

Linear algebra

① Calculation of a basis for ran A and for ker A

$EA =$



$A = A_\beta U$ ← indep. rows

↗ indep. colums

$\operatorname{ran} A = \operatorname{ran} A_\beta$: Basis $\{a_{\beta_1}, \dots, a_{\beta_r}\}$

$\ker A = \ker U = \{x : x_\beta = -U_\nu x_\nu\}$

$x_\nu = e_1, \dots, e_{n-r}$

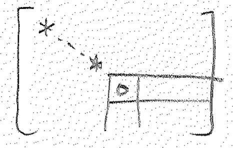
$x_\beta = \dots$

$z_1 \qquad z_{n-r}$

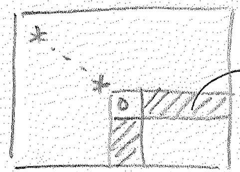
Basis $\{z_1, \dots, z_{n-r}\}$.

② Determining whether a symmetric $H \in \mathbb{R}^{n \times n}$ is p.s.d. / p.d:

$$EHE^T = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$



never p.d.



non zero

never p.s.d.

II. Quadratic optimization

No constraints.

$$\begin{cases} \text{min.} & \frac{1}{2} x^T H x + c^T x + c_0 \\ \text{s.t.} & x \in \mathbb{R}^n \end{cases}$$

H	Conclusion
mot p.s.d.	No optimal solution
p.s.d.	Optimal solution exists iff $-c \in \operatorname{ran} H$. $\hat{x}$ optimal iff $H\hat{x} = -c$.
p.d.	$\exists!$ optimal solution given by $\hat{x} = -H^{-1}c$

Least squares problem.

$$\begin{cases} \text{minimize} & \frac{1}{2} \| Ax - b \|^2 \\ \text{s.t} & x \in \mathbb{R}^n \end{cases}$$

Optimal solution exists.

$\hat{x}$ optimal iff $\underbrace{\boxed{A^T A \hat{x} = A^T b}}_{\text{Normal equations.}}$

15.11

Equality constraints. $(Q): \begin{cases} \text{minimize} & \frac{1}{2} x^T H x + c^T x + c_0 \\ \text{s.t.} & Ax = b . \end{cases}$

Let $\{z_1, \dots, z_k\}$ be a basis for $\ker A$ and let $Z := \begin{bmatrix} z_1 & \cdots & z_k \end{bmatrix}$.

Fix a solution $\bar{x}$ to $Ax = b$. $\left(Ax = b \Leftrightarrow x = \bar{x} + Zv \right)$

If $Z^T H Z$ not p.s.d., then (Q) has no optimal solution.

If $Z^T H Z$ is p.s.d., then:

$$\boxed{\hat{x} \text{ optimal for } (Q)} \Longleftrightarrow \boxed{\begin{array}{l} \exists u \text{ s.t.} \\ \begin{bmatrix} H & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ u \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix} \end{array}} \Longleftrightarrow \boxed{\begin{array}{l} \hat{x} = \bar{x} + Z\hat{v} \\ Z^T H Z \hat{v} = -Z^T (H\bar{x} + c) \end{array}}$$

Lagrange method

Nullspace method

15.12

III Nonlinear optimization

Unconstrained: $\begin{cases} \text{minimize} & f(x) \\ \text{s.t} & x \in \mathbb{R}^n \end{cases}$

Necessity: x_0 local minimizer $\Rightarrow \begin{cases} \nabla f(x_0) = 0 \\ F(x_0) \text{ p.s.d.} \end{cases}$

Sufficiency: $\left.\begin{matrix} \nabla f(x_0) = 0 \\ F(x_0) \text{ p.d.} \end{matrix}\right\} \Rightarrow x_0$ local minimizer.

For $f : \mathbb{R}^n \to \mathbb{R}$ convex: $\boxed{\hat{x} \text{ global minimizer}} \Longleftrightarrow \boxed{\nabla f(\hat{x}) = 0}$

Checking convexity: $C \subset \mathbb{R}^n$ convex set

C has interior points.

Then $\boxed{f : C \to \mathbb{R} \text{ convex}} \Longleftrightarrow \boxed{\forall x \in C,\ F(x) \text{ p.s.d}}$.

Numerical methods (1) Approximating $\hat{x}$ s.t $\nabla f(\hat{x}) = 0$ — Newton's method

(2) Approximating a solution to the Nonlinear Least Squares problem: $\begin{cases} \min & \frac{1}{2}\|h(x)\|^2 \\ \text{s.t} & x \in \mathbb{R}^n \end{cases}$. — Gauss-Newton method.

Constraints

(i) Equality constraints

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & h_1(x)=0 \\ & \vdots \\ & h_m(x)=0. \end{cases}$$

$$\left.\begin{array}{l} x_0 \ \text{local minimizer} \\ \qquad + \\ x_0 \ \text{regular point} \ \ (\nabla h_1(x_0), \dots, \nabla h_m(x_0) \ \text{indep.}) \end{array}\right\} \Longrightarrow \boxed{\begin{array}{l} \exists u \in \mathbb{R}^m \ \text{s.t.} \\ \nabla f(x_0) + u^T \nabla h(x_0) = 0 \end{array}}.$$

Weierstrass' theorem

$$\left.\begin{array}{l} \mathcal{K} \subset \mathbb{R}^n \ \text{compact} \\ f : \mathcal{K} \longrightarrow \mathbb{R} \ \text{continuous} \end{array}\right\} \Rightarrow \boxed{\left\{\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \mathcal{K} \end{array}\right\} \ \text{has a global minimizer}}$$

(ii) Inequality constraints

$$\begin{cases} \text{minimize} & f(x) \\ \text{s.t.} & g_1(x) \le 0 \\ & \vdots \\ & g_m(x) \le 0 \end{cases}$$

$$\left.\begin{array}{l} f, g_1, \dots, g_m \ \text{convex} \\ \qquad + \\ \text{problem is regular.} \end{array}\right\} \Rightarrow \left(\boxed{\hat{x} \ \text{optimal}} \ \text{iff} \ \text{KKT-conditions are satisfied by } \hat{x}. \right)$$

Exam 21 October. Thursday

14:00 – 19:00.

5 questions.

$\geqslant 9$ points $\Rightarrow$ skip Q. 1.

$\geqslant 5$ points $\Rightarrow$ skip Q. 1. (a).

Mark sheet will be with invigilator.

No calculator allowed in the exam.

Formula sheet will be provided in the exam.

Draft of the formula sheet is on the course homepage.