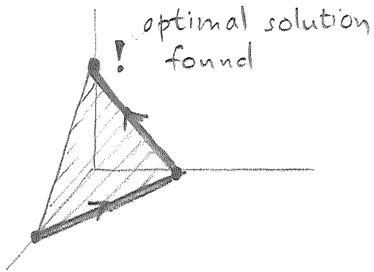
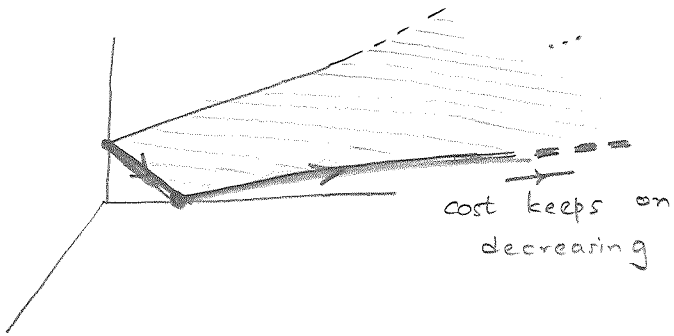
Simplex method

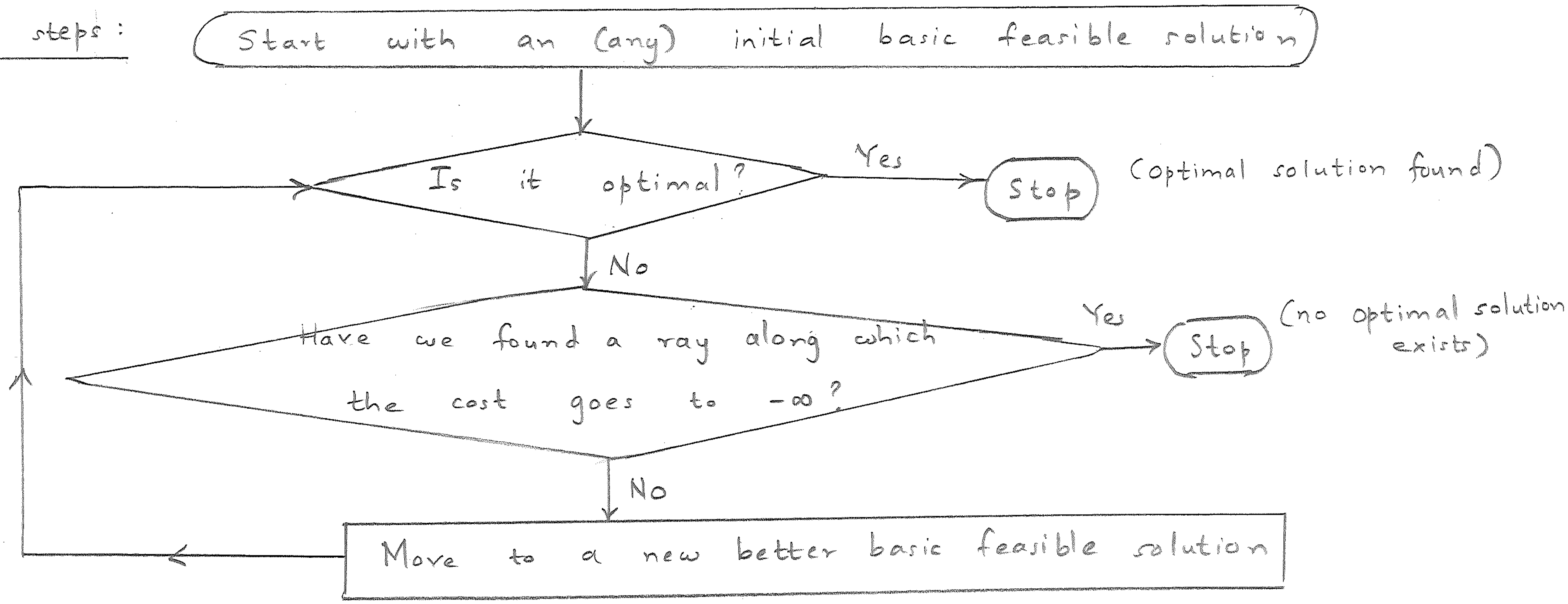
Idea: Proceed from one basic feasible solution to next, while continually decreasing the value of the objective function until
- a minimum is reached
  or
- we discover that there is no optimal solution.





The main point of the simplex method is that it is a sensible way of searching among basic feasible solutions.

Basic steps:

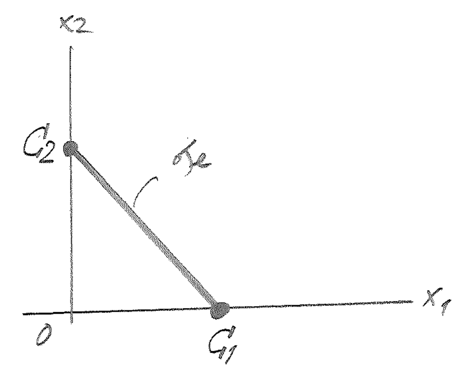


The LP problem is assumed to be in standard form:

$$(P):\ \begin{cases} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{cases} \qquad \begin{array}{l} A \in \mathbb{R}^{m\times n} \\ b \in \mathbb{R}^m \\ c \in \mathbb{R}^n \end{array} \qquad x\ (\text{variable}) \in \mathbb{R}^n$$

with $\boxed{\operatorname{rank} A = m < n.}$

Checking optimality of a basic feasible solution.



Example.

$$(P): \begin{cases} \text{minimize} & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0 . \end{cases}$$

There are 2 basic feasible solutions:

$\beta = (1)$ gives $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv$ corner point C_1.

$\beta = (2)$ gives $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv$ corner point C_2.

Suppose we want to know if C_1 is optimal.

We will express the cost of any other feasible x in terms of the cost of C_1 + left over stuff involving x_N only and see if the left over stuff is always ≥ 0.

Cost of $C_1 = 1 + 2 \cdot 0 = 1$

Cost of any feasible x is

$$\begin{aligned} & x_1 + 2x_2 \\ &= 1 - x_2 + 2x_2 \\ &= 1 + \underbrace{x_2}_{\geq 0} \\ &\geq 1 = \text{cost of } C_1. \end{aligned}$$

So C_1 is optimal.

Based on this example, we now consider the general case.

So suppose we have a basic feasible solution $\bar{x}$ corresponding to the basic tuple $\beta = (\beta_1, \ldots, \beta_m)$.

Then
$$A_\beta \bar{x}_\beta + A_\nu \underbrace{\bar{x}_\nu}_{= 0} = b$$

So β part of this basic feasible solution is $\bar{b}$, given by $A_\beta \bar{b} = b$.

and ν part " " is $\underline{0}$.

The cost of this basic feasible solution is:

$$c^T \bar{x} = c_1 \bar{x}_1 + \dots + c_n \bar{x}_n = c_{\beta_1} \bar{x}_{\beta_1} + \dots + c_{\beta_m} \bar{x}_{\beta_m} + c_{\nu_1} \bar{x}_{\nu_1} + \dots + c_{\nu_{n-m}} \bar{x}_{\nu_{n-m}}$$

$$= c_\beta^T \underbrace{\bar{x}_\beta}_{= \bar{b}} + c_\nu^T \underbrace{\bar{x}_\nu}_{= 0}$$

$$= c_\beta^T \bar{b}.$$

Suppose x is any feasible solution. Know: $Ax = b \rightsquigarrow A_\beta x_\beta + A_\nu x_\nu = b$

$x \geq 0 \rightsquigarrow x_\beta \geq 0$, $x_\nu \geq 0$

Cost of this x is:

$$c^T x = c_\beta^T x_\beta + c_\nu^T x_\nu$$

$$= c_\beta^T A_\beta^{-1} (b - A_\nu x_\nu) + c_\nu^T x_\nu$$

$$= c_\beta^T A_\beta^{-1} b - c_\beta^T A_\beta^{-1} A_\nu x_\nu + c_\nu^T x_\nu$$

$$= \underbrace{c_\beta^T \bar{b}}_{\text{cost of } \bar{x}} + \underbrace{\left(c_\nu - A_\nu^T A_\beta^{-T} c_\beta\right)^T}_{r_\nu} \underbrace{x_\nu}_{\geq 0}$$

Note if r_ν has each component ≥ 0, then $c^T x \geq$ cost of $\bar{x}$ $\forall$ feasible x.

So then $\bar{x}$ is optimal.

r_ν : Reduced costs for nonbasic variables.

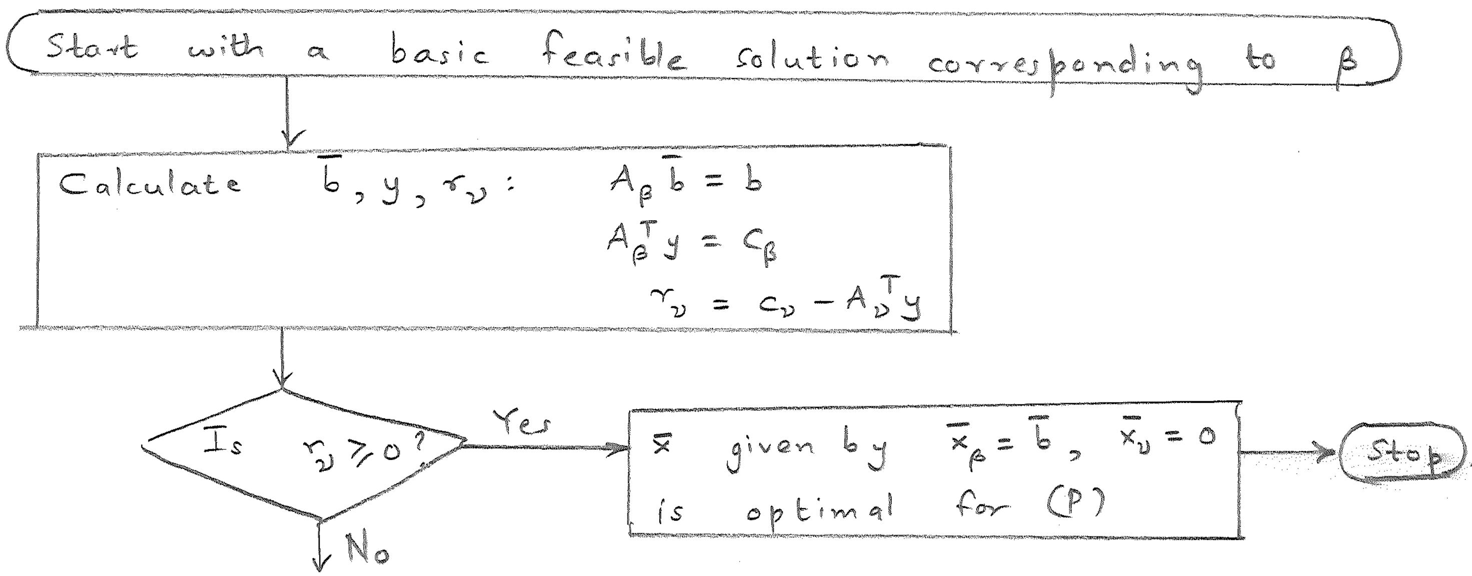
Calculating r_ν : Two steps :

(1) Calculate the 'simplex multipliers vector' y using $A_\beta^T y = c_\beta$.

(note $y = A_\beta^{-T} c_\beta$).

(2) Find $r_\nu = c_\nu - A_\nu^T y$.

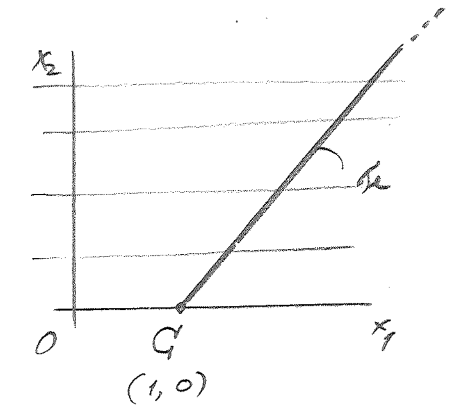
How far have we come in the simplex algorithm? :



Have we found a ray in the feasible set along which the cost goes to $-\infty$?

<u>Example</u>. Consider

$$\begin{cases} \text{minimize} & -2x_2 \\ \text{s.t.} & x_1 - x_2 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0 \end{cases}$$



There is only one basic feasible solution,

corresponding to $\beta = (1)$, namely $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (corner point C).

(With $\beta = (2)$, we get the basic, but not feasible, solution $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.)

Is the reduced cost $r_\nu \geq 0$?

Calculate y first using $A_\beta^T y = c_\beta$ i.e., $1y = 0$, i.e., $y = 0$.

Then $r_\nu = c_\nu - A_\nu^T y = -2 - (-1)(0) = -2 + 0 = -2 < 0$.

So we cannot conclude that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is optimal.

So we hope we can find a better basic feasible solution.

Set $x_\nu = x_2 = t$ and solve for $x_\beta = x_1$ using $x_1 - x_2 = 1$.

We find $x_1 = 1 + t$.

Note that if $t = 0$, we are at C.

If t increases from 0, $x_\nu = x_2 = t > 0$ and

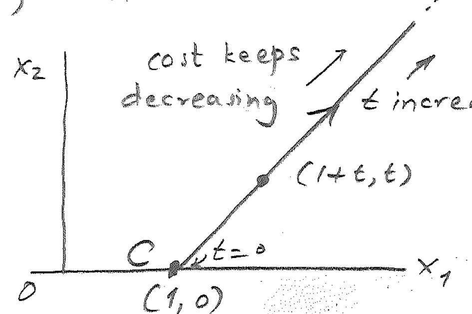
$x_\beta = x_1 = 1 + t > 0$.

So x formed from putting together x_β and x_ν is feasible.

But the cost of this x is:

$$
\begin{aligned}
c^T x &= \text{cost of } C + \text{reduced cost} \cdot x_\nu \\
&= \text{cost of } C + -2 \cdot t \\
&= \text{cost of } C - 2t \\
&\longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty .
\end{aligned}
$$

So we have found a ray (namely $(1+t, t)$ in $\mathbb{R}^2$) along which the cost keeps decreasing.



Thus there is no optimal solution.

Based on this example we now consider the general case.

So let β be the basic tuple and let $\bar{x}$ be the corresponding basic feasible solution for which it is not the case that $r_{\nu_q} \geq 0$.

$$r_\nu = \begin{bmatrix} r_{\nu_1} \\ \vdots \\ \\ r_{\nu_{n-m}} \end{bmatrix} \quad < 0$$

There is an index q s.t. r_{ν_q} is the most negative component of r_ν.

Calculate $\bar{a}_{\nu_q}$ using $A_\beta \, \bar{a}_{\nu_q} = a_{\nu_q}$.

<u>Claim</u>: If $\bar{a}_{\nu_q} \leq 0$, then we have found a ray in Fe along which the cost goes to $-\infty$, and so there is no optimal solution.

<u>Reason</u> : Let $t > 0$.

Set $x_\nu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow q\text{th position} = t e_q \geq 0$.

Now we solve for x_β using $A_\beta x_\beta + A_\nu x_\nu = b$:

$$x_\beta = A_\beta^{-1} \left(b - A_\nu x_\nu \right) = A_\beta^{-1} b - A_\beta^{-1} \begin{array}{c} \phantom{a_{\nu_1}} \quad \swarrow q\text{th} \\ \begin{bmatrix} a_{\nu_1} & \cdots & \vdots & a_{\nu_{n-m}} \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow q$$

$$= \bar{b} - t A_\beta^{-1} a_{\nu_q}$$

$$= \bar{b} - \underbrace{t}_{>0} \underbrace{\bar{a}_{\nu_q}}_{\leq 0} \geq 0 .$$

Then the x formed by putting together this x_β and x_ν is s.t. $x \geq 0$ and $Ax = A_\beta x_\beta + A_\nu x_\nu = b$. So x is feasible.

Its cost is:

$$c^T x = \text{cost of } \bar{x} + r_\nu^T x_\nu$$

$$= \text{cost of } \bar{x} + \begin{array}{c} \phantom{r_{\nu_1}} \quad \swarrow q\text{th} \\ \begin{bmatrix} r_{\nu_1} & \cdots & r_{\nu_{n-m}} \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow q$$

$$= \text{cost of } \bar{x} + \underbrace{t}_{>0} \underbrace{r_{\nu_q}}_{<0}$$

$$\longrightarrow -\infty \quad \text{as} \quad t \longrightarrow +\infty .$$

So there is no optimal solution.

How far have we come in the simplex algorithm?

Start with a basic feasible solution corresponding to β

Calculate $\bar{b}, y, r_\nu$: $A_\beta \bar{b} = b$

$A_\beta^T y = c_\beta$

$r_\nu = c_\nu - A_\nu^T y$

Is $r_\nu \geq 0$?

Yes: $\bar{x}$ given by $\bar{x}_\beta = \bar{b}$, $\bar{x}_\nu = 0$ is optimal for (P) → Stop

No:

Choose index q for which r_{ν_q} is the most negative component of r_ν.

Calculate $\bar{a}_{\nu_q}$ using $A_\beta \bar{a}_{\nu_q} = a_{\nu_q}$.

Is $\bar{a}_{\nu_q} \leq 0$?

Yes: No optimal solution exists (cost along the ray $x_\nu = t e_q$, $x_\beta = \bar{b} - t \bar{a}_{\nu_q}$, $t > 0$ goes to $-\infty$) → Stop

No

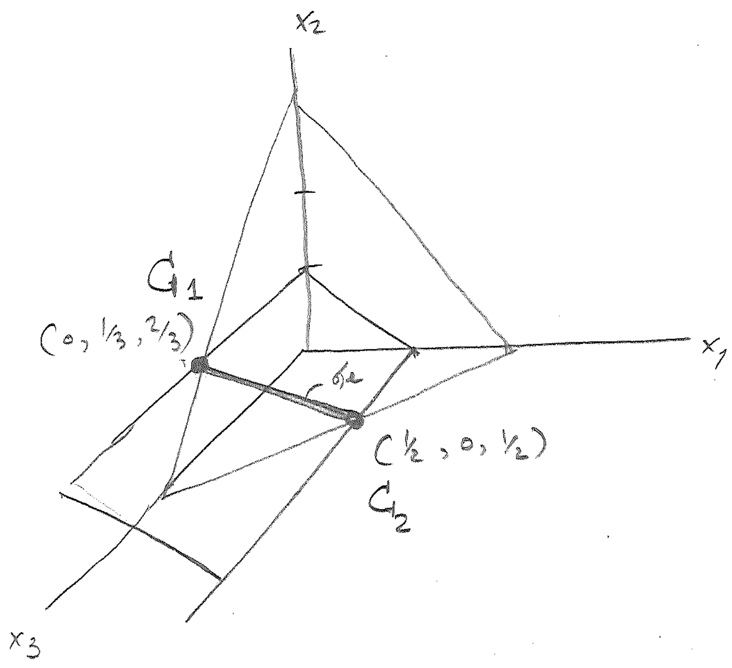
What if it is not the case that $\bar{a}_{\nu_q} \leq 0$,

i.e., some component of $\bar{a}_{\nu_q}$ is strictly positive? $\equiv$ smaller cost

Then we can construct a "better" basic feasible solution by going along the same ray as before, except that now we can't go on forever (we will hit the boundary of $\mathcal{X}$, at which point we stop).

Example:

$$\begin{cases} \text{minimize} & x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 = 1 \\ & 2x_1 + 3x_2 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{cases}$$



Let $\beta = (2, 3)$.

Then the corresponding basic feasible solution is $\begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix}$, so we are at G_1.

$$A_\beta = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}; \quad \begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 1 \\ -1/3 \end{bmatrix}; \quad r_\nu = 0 - \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1/3 \end{bmatrix} = 0 - 1/3 = -1/3 < 0.$$

So we can't conclude that C_1 is optimal.

Have we found a ray? $A_\beta \bar{a}_1 = a_1 \Rightarrow \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \bar{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \bar{a}_1 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$.

Answer: No.

The ray was given by $x_1 = t$

$$\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \bar{b} - t\bar{a}_1 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} - t \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1-2t \\ 2-t \end{bmatrix}.$$

Note that if $t > 0$, we aren't guaranteed that $\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \geqslant 0$.

In fact $x_2 = 0$ at $t = 1/2$ and if $t > 1/2$, $x_2 < 0$.

Also $x_3 = 0$ at $t = 2$ and if $t > 2$, $x_3 < 0$

So the maximum t can be so that x_2, x_3 are both nonnegative is

$$\min\left\{\frac{1}{2}, 2\right\} = \frac{1}{2} =: t_{max}$$

Note that when $t = t_{max}$, $x_2 = 0$.

Also, $x_1 = t_{max} = \frac{1}{2}$ and $x_3 = \frac{1}{3}\left(2 - \frac{1}{2}\right) = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$.

So we have arrived at the new basic feasible solution

$x = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$ corresponding to $\beta_{new} = (\underset{\nu_q = 1}{\overset{2}{\boxed{1}}}, 3)$. (Corner point C_2).

3.15:

$$\text{The cost of } G_2 = \text{cost of } G_1 + r_{\nu_{old}}^T \overset{= x_1}{x_{\nu_{old}}}$$
$$= \text{cost of } G_1 + (-1)(1/2)$$
$$= \text{cost of } G_1 - 1/2$$
$$< \text{cost of } G_1 .$$

So we have come to a new, better basic feasible solution.

Based on this example, we now consider the general case.

So we have a basic feasible solution corresponding to β

for which r_ν was not ≥ 0,

and so we took a q s.t. $r_{\nu q}$ is the most negative component of r_ν

and then we asked: have we found a ray in $\mathcal{F}$ along which the cost keeps decreasing?

So we calculated $\bar{a}_{\nu q}$ using: $A_\beta \bar{a}_{\nu q} = a_{\nu q}$,

but found out that $\bar{a}_{\nu q}$ isn't ≤ 0. Now we see how to construct a better basic feasible solution.

Consider again the ray candidate, given by

$$x_\nu = t e_q$$

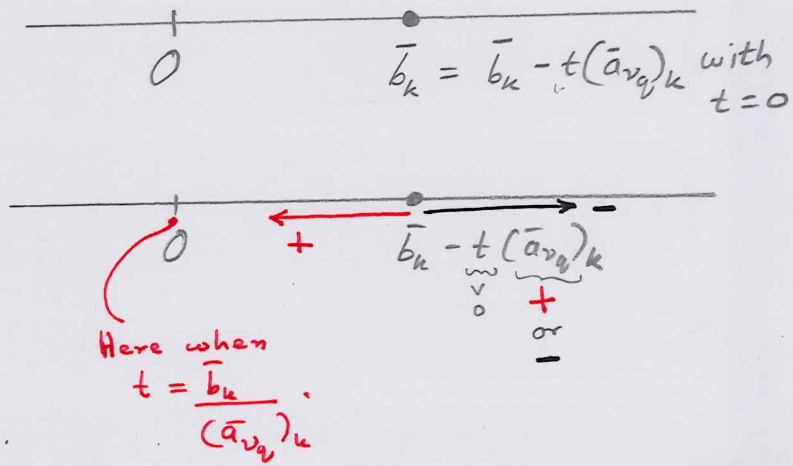
$$x_\beta = \bar{b} - \underbrace{t}_{\geq 0} \underbrace{\bar{a}_{\nu_q}}_{\cancel{\geq 0}} \quad \text{for } t \geq 0 .$$

But now, x_β is not always ≥ 0.

The x obtained this way does satisfy $Ax = b$.

To get the new better basic feasible solution, we take the largest nonnegative t ensuring $x_\beta \geq 0$.

Note that $x_\beta = \bar{b} - t\,\bar{a}_{\nu_q} = \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - t \begin{bmatrix} (\bar{a}_{\nu_q})_1 \\ \vdots \\ (a_{\nu_q})_m \end{bmatrix}$



The largest t can be ensuring that $x_\beta \geq 0$ is thus

$$t_{max} = \min \left\{ \frac{(\bar{b})_k}{(\bar{a}_{\nu_q})_k} : (\bar{a}_{\nu_q})_k > 0 \right\} = \frac{\bar{b}_p}{(\bar{a}_{\nu_q})_p} \geq 0 \quad \text{for some index } p.$$

Then $x_{\beta_p} = 0$.

This x is better than the basic feasible solution $\bar{x}$ we started from:

$$c^T x = c^T \bar{x} + r_\nu^T x_\nu = c^T \bar{x} + r_\nu^T (t_{max}\, e_q) = c^T \bar{x} + \underbrace{\underbrace{r_{\nu_q}}_{<0} \underbrace{t_{max}}_{\geq 0}}_{\leq 0} \leq c^T \bar{x}$$

This x is a new basic feasible solution corresponding to the basic tuple:

$$\beta_{new} = \Big(\beta_1, \dots, \beta_{p-1}, \boxed{\nu_q}, \beta_{p+1}, \dots, \beta_m\Big)$$

β_p $(x_{\beta_p} = 0)$

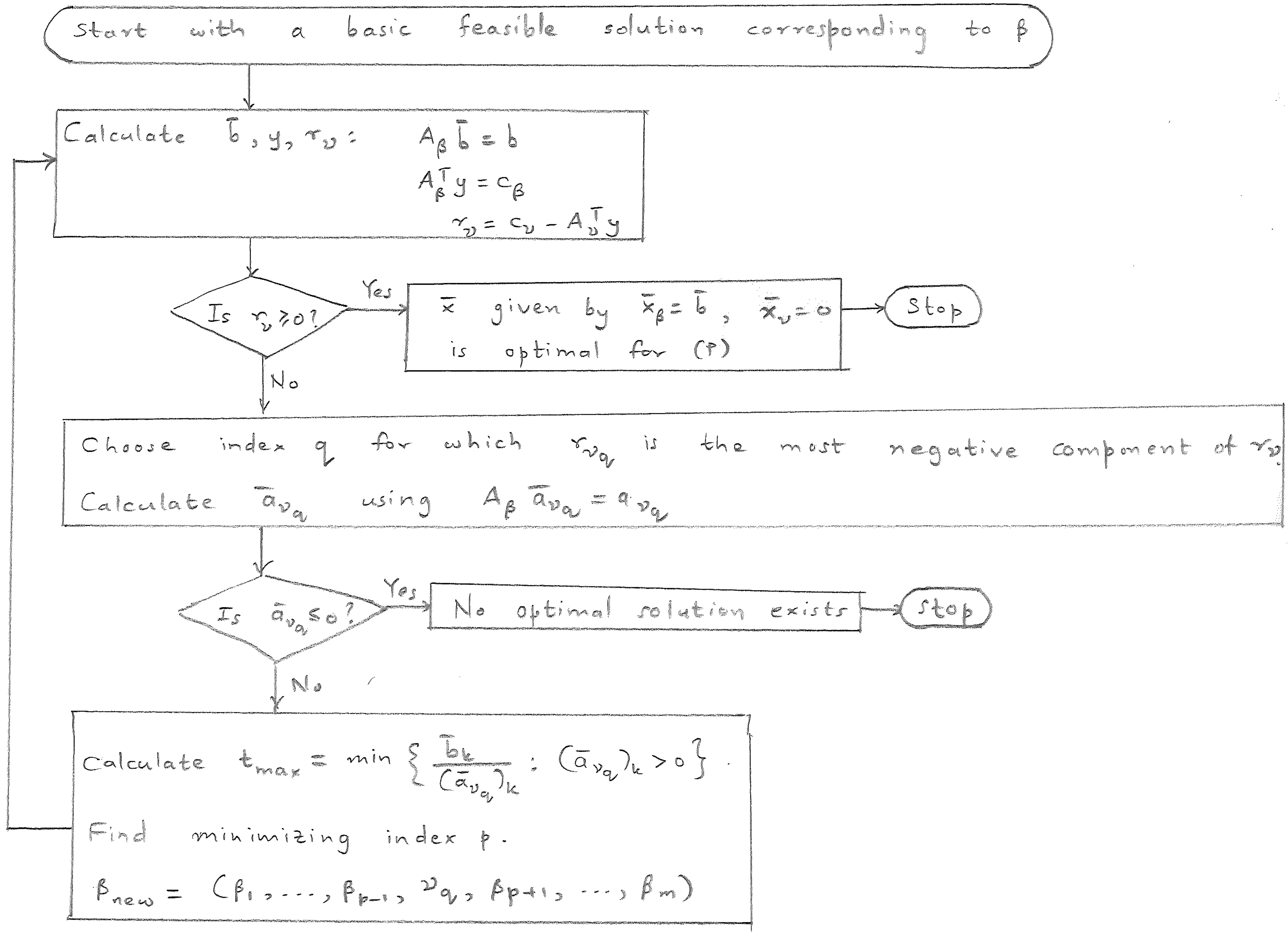
$(x_{\nu_q} = t_{max})$

ν_q

$$\nu_{new} = \Big(\nu_1, \dots, \nu_{q-1}, \boxed{\beta_p}, \nu_{q+1}, \dots, \nu_{n-m}\Big)$$

β_p $(x_{\beta_p} = 0)$

So now we have the complete simplex algorithm.



<u>Homework:</u> Carry it out for the production planning example we had considered.