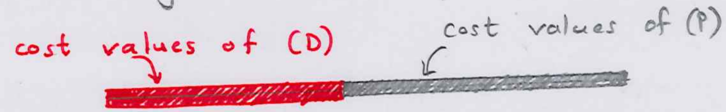
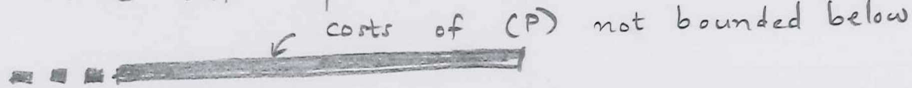
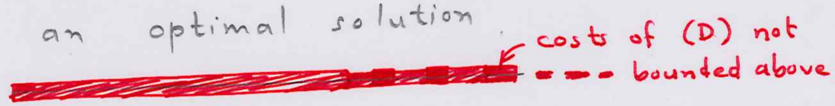
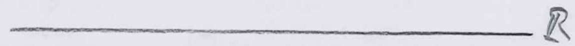
**Theorem** (Duality theorem)

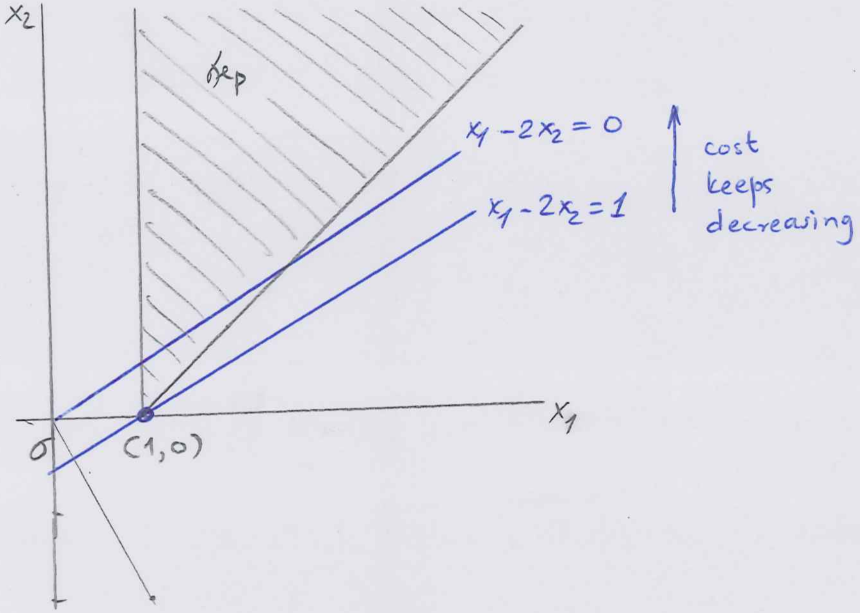
| | $\mathcal{F}_P$ | $\mathcal{F}_D$ | Conclusion |
|---|---|---|---|
| (1) | $\neq \phi$ | $\neq \phi$ | $\exists$ an optimal solution $\hat{x}$ to (P).<br>$\exists$ an optimal solution $\hat{y}$ to (D).<br>$c^T\hat{x} = b^T\hat{y}$<br> |
| (2) | $\neq \phi$ | $= \phi$ | $\forall \rho \in \mathbb{R}$ $\exists$ $x \in \mathcal{F}_P$ s.t. $c^Tx < \rho$.<br>Neither (P) nor (D) have an optimal solution<br> |
| (3) | $= \phi$ | $\neq \phi$ | $\forall \rho \in \mathbb{R}$ $\exists$ $y \in \mathcal{F}_D$ s.t. $b^Ty > \rho$.<br>Neither (P) nor (D) have an optimal solution<br> |
| (4) | $= \phi$ | $= \phi$ | Neither (P) nor (D) have an optimal solution.<br> |

<u>Example</u>

$$\begin{cases} \text{minimize} & x_1 - 2x_2 \\ \text{s.t.} & -x_1 + x_2 \geqslant -1 \\ & x_1 \geqslant 1 \\ & x_1 \geqslant 0 \\ & x_2 \geqslant 0 \end{cases} \qquad c = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

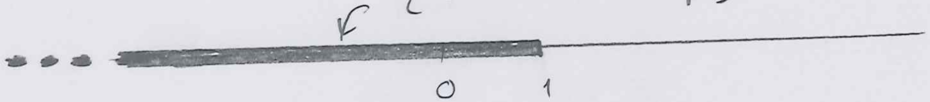
$$\begin{cases} \text{maximize} & -y_1 + y_2 \\ \text{s.t.} & -y_1 + y_2 \leq 1 \\ & y_1 \leq -2 \\ & y_1 \geqslant 0 \\ & y_2 \geqslant 0 \end{cases}$$

$\lightning$ Cannot be satisfied simultaneously




$$\mathcal{X}_D = \emptyset$$

$\{c^T x : x \in \mathcal{X}_P\}$




(P) has no optimal solution.

## Example (continued)

$$(D): \begin{cases} \text{maximize} & -y_1 + y_2 \\ \text{s.t.} & -y_1 + y_2 \leq 1 \\ & y_1 \leq -2 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \end{cases}$$

$$(P): \begin{cases} \text{minimize} & x_1 - 2x_2 \\ \text{s.t.} & -x_1 + x_2 \geq -1 \\ & x_1 \geq 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 . \end{cases}$$

$\equiv$ $\equiv$

$$(\tilde{P}): \begin{cases} \text{minimize} & x_1 - x_2 \\ \text{s.t.} & x_1 - x_2 \geq -1 \\ & -x_1 \geq 2 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

$$c = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

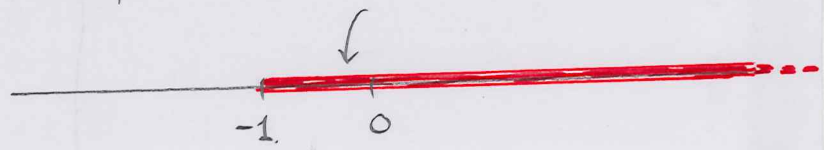
$\xrightarrow{\text{DUAL}}$

$$(\tilde{D}): \begin{cases} \text{maximize} & -y_1 + 2y_2 \\ \text{s.t.} & y_1 - y_2 \leq 1 \\ & -y_1 \leq -1 \\ & y_1 \geq 0 \\ & y_2 \geq 0 . \end{cases}$$

$\mathcal{F}_{\tilde{P}} = \emptyset$.

$\mathcal{F}_{\tilde{D}} \neq \emptyset$

possible costs of $(\tilde{D})$




$(\tilde{D})$ has no optimal solution.

<u>Corollary</u>. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

| $x$ is optimal for (P) and $y$ is optimal for (D) | $\iff$ | $x$ is feasible for (P), $y$ is feasible for (D), and $c^T x = b^T y$ |
|---|---|---|

<u>Proof</u> $\boxed{\Rightarrow}$: $c^T x = c^T \hat{x} = b^T \hat{y} = b^T y$.

$\boxed{\Leftarrow}$: Let $\tilde{x}$ be feasible for (P).

Then $c^T \tilde{x} \geqslant b^T y = c^T x$. So $x$ is optimal for (P)

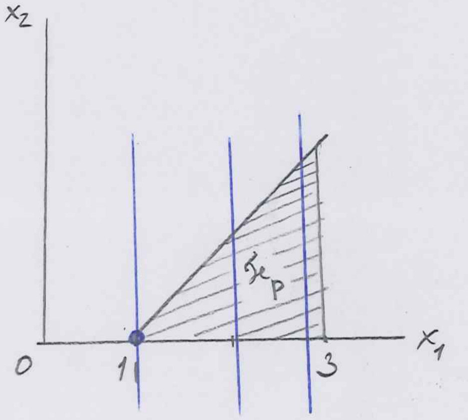
Let $\tilde{y}$ be feasible for (D)

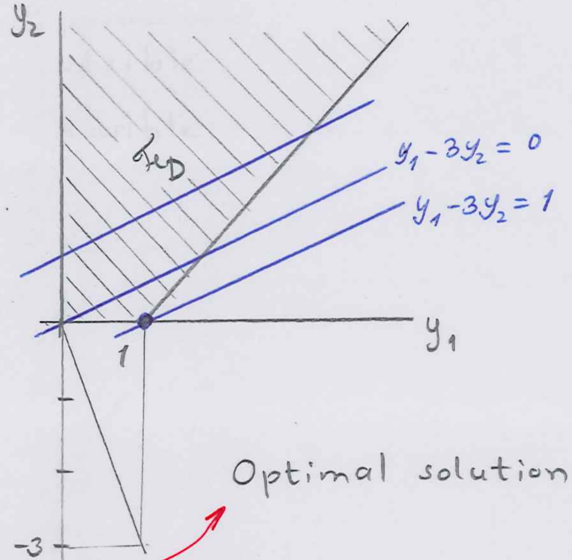
Then $b^T \tilde{y} \leq c^T x = b^T y$. So $y$ is optimal for (D). $\square$

<u>Example</u>

$$(P):\begin{cases}\text{minimize} & x_1\\ \text{s.t} & x_1 - x_2 \geq 1\\ & -x_1 \geq -3\\ & x_1 \geq 0\\ & x_2 \geq 0\end{cases}$$

$$(D):\begin{cases}\text{maximize} & y_1 - 3y_2\\ \text{s.t} & y_1 - y_2 \leq 1\\ & -y_1 \leq 0\\ & y_1 \geq 0\\ & y_2 \geq 0 .\end{cases}$$




Optimal solution $x = \begin{bmatrix}1\\0\end{bmatrix}$

Optimal solution $y = \begin{bmatrix}1\\0\end{bmatrix}$.

$$r = Ax - b = \begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} - \begin{bmatrix}1\\-3\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$$

$$s = c - A^T y = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$x_1 \cdot s_1 = 1.0 = 0$
$x_2 \cdot s_2 = 0.1 = 0$.

$y_1 \cdot r_1 = 1.0 = 0$
$y_2 \cdot r_2 = 0.2 = 0$.

<u>Corollary</u> (Complimentarity theorem)

Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. Define $r := Ax - b$, $s := c - A^T y$. Then:

$$\boxed{\begin{array}{l} x \text{ is optimal for } (P) \\ \text{and} \\ y \text{ is optimal for } (D) \end{array}} \iff \boxed{\begin{array}{lccl} x_i \geq 0 & & x_i s_i = 0 & i = 1, \ldots, n \\ y_j \geq 0 & \text{and} & & \\ r_j \geq 0 & & y_j \cdot r_j = 0 & j = 1, \ldots, m \\ s_i \geq 0 & & & \end{array}}$$

<u>Proof</u>. $\boxed{\Rightarrow}$: $x \in \mathcal{F}_P \Rightarrow x \geq 0$, $r \geq 0$.

$y \in \mathcal{F}_D \Rightarrow y \geq 0$, $s \geq 0$.

$c^T x = b^T y$.

$$\begin{aligned} 0 = c^T x - b^T y = x^T c - y^T b &= x^T(s + A^T y) - y^T(Ax - r) \\ &= x^T s + \cancel{x^T A^T y} - \cancel{y^T A x} + y^T r \\ &= \sum_{i=1}^{n} \underbrace{x_i s_i}_{\geq 0} + \sum_{j=1}^{m} \underbrace{y_j r_j}_{\geq 0}. \end{aligned}$$

So $x_i s_i = 0$ $(i = 1, \ldots, n)$ and $y_j r_j = 0$ $(j = 1, \ldots, m)$.

5.5

$\boxed{\Leftarrow}$:

$$\left.\begin{array}{ll} x_i \geq 0 & \\ r_j \geq 0 & \end{array}\right\} \forall\, i,j \;\Rightarrow\; x \in \mathcal{F}_P$$

$$\left.\begin{array}{ll} y_j \geq 0 & \\ s_i \geq 0 & \end{array}\right\} \forall\, i,j \;\Rightarrow\; y \in \mathcal{F}_D .$$

$$\begin{aligned} 0 = \sum_{i=1}^{n} x_i s_i + \sum_{j=1}^{m} y_j r_j = x^T s + y^T r &= x^T(c - A^T y) + y^T(Ax - b) \\ &= x^T c - \cancel{x^T A^T y} + \cancel{y^T A x} - y^T b \\ &= x^T c - y^T b \\ &= c^T x - b^T y \end{aligned}$$

So $c^T x = b^T y$.

If $\tilde{x} \in \mathcal{F}_P$, then $c^T \tilde{x} \geq b^T y = c^T x$. So $x$ is optimal for (P)

If $\tilde{y} \in \mathcal{F}_D$, then $b^T \tilde{y} \leq c^T x = b^T y$. So $y$ is optimal for (D).

$\square$

## Dual to LP problem in general form:

**Example**

$$(P): \begin{cases} \text{minimize} & 2x_1 + 3x_2 \\ \text{s.t.} & 4x_1 + 5x_2 \geq 6 \\ & 7x_1 + 8x_2 = 9 \\ & x_1 \geq 0 . \end{cases}$$

First convert this into canonical form.

$$\begin{cases} \text{minimize} & 2x_1 + 3x_2 \\ \text{s.t} & 4x_1 + 5x_2 \geq 6 \\ & 7x_1 + 8x_2 \geq 9 \\ & 7x_1 + 8x_2 \leq 9 \\ & x_1 \geq 0 \\ & x_2 \text{ free} \end{cases}$$

Write $x_2 = v_2 - v_3$, $v_2 \geq 0$, $v_3 \geq 0$.

$$(\tilde{P}): \begin{cases} \text{minimize} & 2x_1 + 3v_2 - 3v_3 \\ \text{s.t.} & 4x_1 + 5v_2 - 5v_3 \geq 6 \\ & 7x_1 + 8v_2 - 8v_3 \geq 9 \\ & -7x_1 - 8v_2 + 8v_3 \geq -9 \\ & x_1 \geq 0 \\ & v_2 \geq 0 \\ & v_3 \geq 0 \end{cases}$$

$$c = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 5 & -5 \\ 7 & 8 & -8 \\ -7 & -8 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 9 \\ -9 \end{bmatrix}$$

The dual to $(\tilde{P})$ is:

$$(\tilde{D}): \begin{cases} \text{maximize} & 6y_1 + 9u_2 - 9u_3 \\ \text{s.t.} & 4y_1 + 7u_2 - 7u_3 \le 2 \\ & 5y_1 + 8u_2 - 8u_3 \le 3 \\ & -5y_1 - 8u_2 + 8u_3 \le -3 \\ & y_1 \ge 0 \\ & u_2 \ge 0 \\ & u_3 \ge 0 \end{cases}$$

i.e., 
$$(\tilde{D}): \begin{cases} \text{maximize} & 6y_1 + 9(u_2 - u_3) \\ \text{s.t.} & 4y_1 + 7(u_2 - u_3) \le 2 \\ & 5y_1 + 8(u_2 - u_3) = 3 \\ & y_1 \ge 0 \\ & u_2 \ge 0 \\ & u_3 \ge 0. \end{cases}$$

This $(\tilde{D})$ is equivalent to:

$$(D): \begin{cases} \text{maximize} & 6y_1 + 9y_2 \\ \text{s.t.} & 4y_1 + 7y_2 \le 2 \\ & 5y_1 + 8y_2 = 3 \\ & y_1 \ge 0 \\ & y_2 \text{ free}. \end{cases}$$

5.9

<u>In general:</u>

$$(P): \begin{cases} \text{minimize} & c_1^T x_1 + c_2^T x_2 \\ \text{s.t} & A_{11}x_1 + A_{12}x_2 \geq b_1 \\ & A_{21}x_1 + A_{22}x_2 = b_2 \\ & x_1 \geq 0 \\ & x_2 \text{ free.} \end{cases}$$

$x_1, x_2$ vector variables of sizes $n_1, n_2$.

First convert to canonical form:

$$\begin{cases} \text{minimize} & c_1^T x_1 + c_2^T x_2 \\ \text{s.t.} & A_{11}x_1 + A_{12}x_2 \geq b_1 \\ & A_{21}x_1 + A_{22}x_2 \geq b_2 \\ & -A_{21}x_1 - A_{22}x_2 \geq -b_2. \\ & x_1 \geq 0 \\ & x_2 \text{ free} \end{cases}$$

Write $x_2 = v_2 - v_3$, $v_2 \geq 0$, $v_3 \geq 0$.

$$(\tilde{P}): \begin{cases} \text{minimize} & c_1^T x_1 + c_2^T v_2 - c_2^T v_3 \\ \text{s.t.} & A_{11}x_1 + A_{12}v_2 - A_{12}v_3 \geq b_1 \\ & A_{21}x_1 + A_{22}v_2 - A_{22}v_3 \geq b_2 \\ & -A_{21}x_1 - A_{22}v_2 + A_{22}v_3 \geq -b_2 \\ & x_1 \geq 0 \\ & v_2 \geq 0 \\ & v_3 \geq 0. \end{cases}$$

$$(\tilde{P}): \begin{cases} \text{minimize} & C^T X \\ \text{s.t.} & A X \geq B \\ & X \geq 0 \end{cases} \qquad \text{where} \quad C = \begin{bmatrix} c_1 \\ c_2 \\ -c_2 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ v_2 \\ v_3 \end{bmatrix} \quad A = \begin{bmatrix} A_{11} & A_{12} & -A_{12} \\ A_{21} & A_{22} & -A_{22} \\ -A_{21} & -A_{22} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ -b_2 \end{bmatrix}$$

The dual is:

$$(\tilde{D}): \begin{cases} \text{maximize} & b_1^T y_1 + b_2^T u_2 - b_2^T u_3 \\ \text{s.t.} & A_{11}^T y_1 + A_{21}^T u_2 - A_{21}^T u_3 \leq c_1 \\ & A_{12}^T y_1 + A_{22}^T u_2 - A_{22}^T u_3 \leq c_2 \\ & -A_{12}^T y_1 - A_{22}^T u_2 + A_{22}^T u_3 \leq -c_2 \\ & y_1 \geq 0 \\ & u_2 \geq 0 \\ & u_3 \geq 0 \end{cases}$$

i.e.,

$$\text{i.e., } (\tilde{D}): \begin{cases} \text{maximize} & b_1^T y_1 + b_2^T (u_2 - u_3) \\ \text{s.t} & A_{11}^T y_1 + A_{21}^T (u_2 - u_3) \leq c_1 \\ & A_{12}^T y_1 + A_{22}^T (u_2 - u_3) = c_2 \\ & y_1 \geq 0 \\ & u_2 \geq 0 \\ & u_3 \geq 0. \end{cases}$$

This is equivalent to:

$$(D): \begin{cases} \text{maximize} & b_1^T y_1 + b_2^T y_2 \\ \text{s.t.} & A_{11}^T y_1 + A_{21}^T y_2 \leq c_1 \\ & A_{12}^T y_1 + A_{22}^T y_2 = c_2 \\ & y_1 \geq 0 \\ & y_2 \text{ free.} \end{cases}$$

<u>Thus</u>: The dual of

$$(P): \left\{ \begin{array}{lll} \text{minimize} & c_1^T x_1 + c_2^T x_2 \\ \text{s.t.} & A_{11} x_1 + A_{12} x_2 \geqslant b_1 \\ & A_{21} x_1 + A_{22} x_2 = b_2 \\ & x_1 \geqslant 0 \\ & x_2 \text{ free} \end{array} \right\}$$

is

$$(D): \left\{ \begin{array}{lll} \text{maximize} & b_1^T y_1 + b_2^T y_2 \\ \text{s.t} & A_{11}^T y_1 + A_{21}^T y_2 \leqslant c_1 \\ & A_{12}^T y_1 + A_{22}^T y_2 = c_2 \\ & y_1 \geqslant 0 \\ & y_2 \text{ free} \end{array} \right\}.$$

<u>Special case</u>: Dual of LP problem in <u>standard form</u>

$$(P): \left\{ \begin{array}{ll} \text{minimize} & c^T x \\ \text{s.t.} & Ax = b \\ & x \geqslant 0. \end{array} \right.$$

| | |
|---|---|
| $x = x_1$ | $x_2$ empty. |
| $A_{21} = A$ | $\left.\begin{array}{l} c_2 \\ b_1,\ A_{11}, A_{12} \\ A_{22} \end{array}\right\}$ empty. |
| $b_2 = b$. | |
| $c_1 = c$ | |

So the dual is:

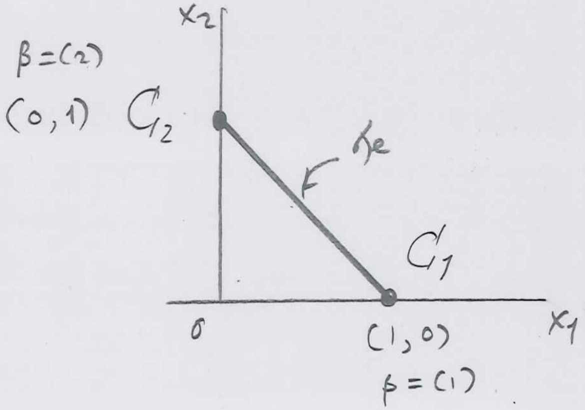
$$(D): \left\{ \begin{array}{ll} \text{maximize} & b^T y \\ \text{s.t.} & A^T y \leqslant c \end{array} \right.$$

(Notice that (D) is not in standard form. This was the reason for starting with the canonical form, which is nicer.)

## Example

(P) : $\begin{cases} \text{minimize} & x_1 + 2x_2 \\ \text{s.t.} & x_1 + x_2 = 1 \\ & x_1 \geqslant 0 \\ & x_2 \geqslant 0 . \end{cases}$

(D) : $\begin{cases} \text{maximize} & y \\ \text{s.t.} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} y \leq \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{cases}$ i.e., $y \leq 1$, $y \leq 2$




(D) : $\begin{cases} \text{maximize} & y \\ \text{s.t.} & y \leq 1 . \end{cases}$

$y = 1$ is an optimal solution.

## $\mathcal{C}_1$ is optimal.

Simplex multipliers vector $y$ : $A_\beta^T y = c_\beta$

$1 \cdot y = 1$

$\Rightarrow y = 1$.

Same ! (This is not a coincidence.)

Reduced costs of the nonbasic variables :

$$r_\nu = c_\nu - A_\nu^T y = 2 - 1 \cdot 1 = 1 \geqslant 0 .$$

So $\mathcal{C}_1 = (1, 0)$ is optimal.

Suppose we have solved the LP problem (P) in standard form using the simplex method, and that the algorithm was terminated since

$r_\nu \geq 0$ (i.e., an optimal solution is found: the last basic feasible solution $\bar{x}$ corresponding to the basic tuple $\beta$ is optimal.)

<u>Claim</u>. The last simplex multiplier vector $y$ is optimal for the dual problem (D)

<u>Proof</u>. <u>Recall</u>: $A_\beta^T y = c_\beta$.

$$c^T \bar{x} = c_\beta^T \bar{b} + c_\nu^T 0 \qquad (\bar{b} : A_\beta \bar{b} = b)$$
$$= y^T A_\beta \bar{b}$$
$$= y^T b = b^T y.$$

<u>Question</u>: Is $y$ feasible for (D)? I.e., is $A^T y \leq c$?

$$r_\nu = c_\nu - A_\nu^T y \geq 0.$$

So $\left.\begin{array}{l} A_\nu^T y \leq c_\nu \\ A_\beta^T y = c_\beta \end{array}\right\} \Rightarrow A^T y \leq c.$

$$\left.\begin{array}{c} a_{\nu_1}^T \\ \vdots \\ a_{\nu_{n-m}}^T \end{array}\right\} A_\nu^T \qquad \left.\begin{array}{c} a_{\beta_1}^T \\ \vdots \\ a_{\beta_m}^T \end{array}\right\} A_\beta^T \qquad \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} y \leq c$$

<u>Answer</u>: Yes.

So $y$ is optimal for (D):

If $\tilde{y}$ is feasible for (D), then

$$b^T\tilde{y} \leq c^T\bar{x} = b^T y. \qquad \square$$

Proof of the duality theorem uses Farkas's Lemma.

This will also be used later while deriving the necessity of KKT-conditions for nonlinear programming problems.

<u>Farkas's lemma</u>

$p_1, \dots, p_m \in \mathbb{R}^n$
$q \in \mathbb{R}^n$.

Then one and exactly one of the following occurs:

(F1) $\exists\, y_1 \geq 0, \dots, y_m \geq 0$ s.t. $q = y_1 p_1 + \dots + y_m p_m$.

(F2) $\exists\, x \in \mathbb{R}^n$ s.t. $q^T x < 0$ and $p_1^T x \geq 0, \dots, p_m^T x \geq 0$