KTH Mathematics

SF2812 Applied linear optimization, final exam
Wednesday January 16 2008 14.00–19.00
Brief solutions

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1. (a) The primal-dual nonlinear equation is given by

$$Ax = b,$$
$$A^Ty + s = c,$$
$$XSe = \sigma\mu e,$$

where $e = (1\ 1\ \dots 1)^T$ and $\mu = (x^Ts)/n = 1.54$ for some $\sigma \in [0, 1]$. With $X = \mathrm{diag}(x)$ and $S = \mathrm{diag}(s)$ the linear system of equations may be written

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^Ty + s - c \\ XSe - \sigma\mu e \end{pmatrix}.$$

We may for example let $\sigma = 0.1$. Insertion of numerical values gives

$$\begin{pmatrix} 2 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta x_5 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta y_3 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \\ \Delta s_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1.546 \\ -0.846 \\ -0.646 \\ -0.046 \\ -3.846 \end{pmatrix}.$$

(b) If we let $\alpha_{\max}$ be the maximum value of α for which $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s \geq 0$, we must have $\alpha < \alpha_{\max}$. Ideally we would want steplength one. One (crude) choice would be $\alpha = \min\{1, 0.99\alpha_{\max}\}$, and then let

$$x = x + \alpha\Delta x, \quad y = y + \alpha\Delta y, \quad s = s + \alpha\Delta s.$$

2. (a) We have x^* nonnegative with $Ax^* = b$ and

$$A_+ = \begin{pmatrix} 4 & -1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

We see that A_+ has a leading nonsingular submatrix of dimension 2×2. Hence, A_+ has full column rank. It follows that x^* is a basic feasible solution.

(b) First (i). Let $x_B = (x_1\ x_2\ x_3)^T$. Then $x_B = (0\ 1\ 2)^T$. Compute y from $B^Ty = c_B$ and let $s_N = c_N - N^Ty$. We obtain

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ -1 \end{pmatrix}, \quad \text{which gives} \quad y = \begin{pmatrix} 1 \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}, \begin{pmatrix} s_4 \\ s_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}.$$

Hence, since $s_N \geq 0$, the simplex method shows that x^* is optimal.

Now (ii). Let $x_B = (x_2\ x_3\ x_4)^T$. Then $x_B = (1\ 2\ 0)^T$. Compute y from $B^Ty = c_B$ and let $s_N = c_N - N^Ty$. We obtain

$$\begin{pmatrix} 4 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ -1 \end{pmatrix}, \quad \text{which gives} \quad y = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Since $s_1 < 0$, x_1 will enter the basis. We obtain the change in the basic variables from $Bp_B = -A_1$, i.e.,

$$\begin{pmatrix} 4 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} -3 \\ -10 \\ -2 \end{pmatrix}.$$

Since $x_4 = 0$, it follows that x_4 leaves the basis, and the new basic variables are $x_B = (x_1\ x_2\ x_3)^T$, which has been covered in (i).

Finally (iii). Let $x_B = (x_2\ x_3\ x_5)^T$. Then $x_B = (1\ 2\ 0)^T$. Compute y from $B^Ty = c_B$ and let $s_N = c_N - N^Ty$. We obtain

$$\begin{pmatrix} 4 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \\ 0 \end{pmatrix}, \quad \text{which gives} \quad y = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Hence, since $s_N \geq 0$, the simplex method shows that x^* is optimal.

Consequently, AF was right in that x^* is optimal. By the simplex method, he could have obtained the final basis as $x_B = (x_1\ x_2\ x_3)^T$ or $x_B = (x_2\ x_3\ x_5)^T$.

3. (a) For $u = 1$, the resulting Lagrangian relaxed problem becomes

$$(IP_1) \quad \begin{array}{ll} \text{minimize} & -2x_1 - 1x_2 - 3x_3 \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & x_j \in \{0, 1\}, \quad j = 1, \dots, n. \end{array}$$

By enumeration, we find two optimal solutions, $x(1) = (1\ 1\ 0)^T$ and $x(1) = (0\ 0\ 1)^T$.

(b) If $x(1)$ is an optimal solution to the Lagrangian relaxed problem for $u = 1$, a subgradient is given by $3x_1(1) + 6x_2(1) + 7x_3(1) - 8$. Hence, $x(1) = (1\ 1\ 0)^T$ gives a subgradient $s_1 = 1$ and $x(1) = (0\ 0\ 1)^T$ gives a subgradient $s_2 = -1$.

(c) Since $0 = 1/2s_1 + 1/2s_2$, the zero vector is a subgradient to $\varphi(u)$ at $u = 1$. Hence, $u = 1$ is an optimal solution to the dual problem.

4. (See the course material.)

5. The suggested initial extreme points $v_1 = (-1\ \ 1\ -1\ \ 1)^T$ and $v_2 = (-1\ \ 1\ \ 1\ -1)^T$ give the initial basis matrix

$$B = \begin{pmatrix} -1 & 7 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (1\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (-2\ -2)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 7 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

By forming $c^T - y_1 A = (1\ -1\ \ 1\ -1)$ we obtain the subproblem

$$\begin{array}{lll} 2+ & \text{minimize} & x_1 - x_2 + x_3 - x_4 \\ & \text{subject to} & -1 \le x_j \le 1, j = 1, \dots, 4. \end{array}$$

The resulting optimal solution gives a new extreme point $v_3 = (-1\ 1\ -1\ 1)^T$ with reduced cost -2. The corresponding column in the master problem is $(5\ 1)^T$, and we obtain

$$p_B = -B^{-1} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = -\begin{pmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{pmatrix}.$$

By considering the step from α_B along p_B and requiring nonnegativity, we obtain the maximum steplength as $3/8$, and α_2 leaves the basis. Hence, α_3 replaces α_2 as basic variable.

The basic variables are now given by

$$\begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_3) = (-2\ -4)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{7}{3} \end{pmatrix}.$$

By forming $c^T - y_1 A = (2/3\ -1/3\ \ 4/3\ \ 0)$ we obtain the subproblem

$$\begin{array}{lll} \frac{7}{3}+ & \text{minimize} & \frac{2}{3}x_1 - \frac{1}{3}x_2 + \frac{4}{3}x_3 \\ & \text{subject to} & -1 \le x_j \le 1, j = 1, \dots, 4. \end{array}$$

The resulting optimal solutions are v_1 and v_3, which both give reduced cost 0. Hence, we have found an optimal solution to the original problem. The solution is given by

$$v_1\alpha_1 + v_3\alpha_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \frac{2}{3} + \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \frac{3}{5} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ \frac{1}{3} \end{pmatrix}.$$

Note: This particular problem may be simplified further, since it is a continuous knapsack problem. By noting that in the subproblem, if we denote the optimal solution of the subproblem by $x(y_1)$, we obtain $x_i(y_1) = -1$ if $c_i - ya_i < 0$ and $x_i(y_1) = -1$ if $c_i - ya_i > 0$. Hence, if we order the ratios c_i/a_i in decreasing order, we obtain $c_3/a_3 = 1$, $c_2/a_2 = 1/2$, $c_4/a_4 = -1/3$, $c_1/a_1 = -1$. Thus, we may start with $y_1 < -1$ for which $x(y_1)$ gives the maximum value of the constraint $-x_1 + 2x_2 + x_3 - 3x_4 - 1$ in the interval $1- \le x_i \le 1$, $i = 1, \dots, 4$. We may then increase y_1 until we reach one point among -1, $-1/3$, $1/2$ and 1 at which passing this point with y_1 makes the constraint $-x_1(y_1) + 2x_2(y_1) + x_3(y_1) - 3x_4(y_1) - 1$ switch from being positive to being negative. This is $y_1 = -1/3$ in this case, as was concluded in the final master problem. Then the variable that switches at this point may be assigned a value in the interval that makes the constraint satisfied. Rather than solve a sequence of master problems, we can increase y_1 over the finite set of points, and need then only solve one subproblem to get the appropriate linear combination.