KTH Mathematics

SF2812 Applied linear optimization, final exam
Monday October 20 2008 8.00–13.00
Brief solutions

1. (a) There is at least one optimal solution, which is integer valued. However, if the optimal solution is nonunique, there will also be noninteger optimal solutions.

(b) Since $\widehat{X}$ is nonnegative, summation of rows and columns of $\widehat{X}$ shows that $\widehat{X}$ is feasible. If we let the matrix S denote the dual slacks, i.e., $s_{ij} = c_{ij} - \widehat{u}_i - \widehat{v}_j$, then

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 2 & 2 & 0 & 0 \end{pmatrix}.$$

Consequently, S has nonnegative components. In addition, complementarity holds, since $\widehat{x}_{ij} s_{ij} = 0$, $i = 1, \dots, 3$, $j = 1, \dots, 4$. This means that we have optimal solutions to the two problems.

(c) The nonzero components of the given U correspond to strictly positive components of $\widehat{X}$. By the properties of U, it follows that $\widehat{X} + \alpha U$ is optimal as long as $\widehat{X} + \alpha U$ is nonnegative. The most limiting positive and negative values of α are 0.5 and −1.5 respectively. These values correspond to two integer valued optimal solutions:

$$\widehat{X} - 1.5U = \begin{pmatrix} 8 & 0 & 0 & 2 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 3 & 7 \end{pmatrix} \quad \text{and} \quad \widehat{X} + 0.5U = \begin{pmatrix} 8 & 2 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 0 & 0 & 1 & 9 \end{pmatrix}.$$

(In this case, $\widehat{X} - 0.5U$ is also an integer valued optimal solution.)

(d) Since $\widehat{X}$ is not an extreme point, it is not provided as a solution by the simplex method.

2. (See the course material.)

3. (a) With $X = \mathrm{diag}(x)$ and $S = \mathrm{diag}(s)$, the linear system of equations takes the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta s \end{pmatrix} = - \begin{pmatrix} Ax - b \\ A^T y + s - c \\ XSe - \mu e \end{pmatrix}.$$

Insertion of numerical values gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \\ \Delta y_1 \\ \Delta y_2 \\ \Delta s_1 \\ \Delta s_2 \\ \Delta s_3 \\ \Delta s_4 \end{pmatrix} = \begin{pmatrix} -4 \\ -12 \\ -3 \\ -2 \\ 1 \\ 0 \\ -5 \\ -1 \\ -1 \\ 0 \end{pmatrix}.$$

(b) If we compute $\alpha_{\max}$ as the largest step α for which $x + \alpha\Delta x \geq 0$ and $s + \alpha\Delta s \geq 0$ we obtain $\alpha_{\max} = 10/21$. As $\alpha_{\max} < 1$ we cannot accept the unit step. If we let $\alpha = 0.99\alpha_{\max}$ the new iterates become $x + \alpha\Delta \approx (1.7171\ \ 2.2714\ \ 0.0200\ \ 0.9057)^T$, $y + \alpha\Delta y \approx (-0.8486\ \ 0.1886)^T$, and $s + \alpha\Delta s \approx (2.2457\ 0.5286\ \ 1.7543\ \ 1.0943)^T$.

4. (a) We may rewrite the linear program as

$$(LP) \quad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & x_i k + l + z \geq y_i, \quad i = 1, \dots, m, \\ & -x_i k - l + z \geq -y_i, \quad i = 1, \dots, m. \end{array}$$

The dual may for example be derived via Lagrangian relaxation. For nonnegative Lagrange multipliers $u \in I\!R^m$ and $v \in I\!R^m$ we obtain

$$\text{minimize} \quad z - \textstyle\sum_{i=1}^m u_i(x_i k + l + z - y_i) - \sum_{i=1}^m v_i(-x_i k - l + z + y_i),$$

which may be rewritten as

$$\textstyle\sum_{i=1}^m y_i u_i - \sum_{i=1}^m y_i v_i + \quad \text{minimize} \quad \begin{array}{l} \{(-\sum_{i=1}^m x_i u_i + \sum_{i=1}^m x_i v_i)k \\ +(-\sum_{i=1}^m u_i + \sum_{i=1}^m v_i)l \\ +(1 - \sum_{i=1}^m u_i - \sum_{i=1}^m v_i)z\}. \end{array}$$

The dual (DLP) then becomes

$$(DLP) \quad \begin{array}{ll} \text{maximize} & \sum_{i=1}^m y_i(u_i - v_i) \\ \text{subject to} & \sum_{i=1}^m x_i(u_i - v_i) = 0, \\ & \sum_{i=1}^m (u_i - v_i) = 0, \\ & \sum_{i=1}^m (u_i + v_i) = 1, \\ & u_i \geq 0, \quad i = 1, \dots, m, \\ & v_i \geq 0, \quad i = 1, \dots, m. \end{array}$$

(b) We need to show that (LP) has an optimal solution with at least three active constraints, corresponding to at least three different points. Basically, (LP) is a three-dimensional problem and hence an extreme point has at least three active constraints. Note that in (LP), $-z \leq kx_i + l - y_i \leq z$, $i = 1, \dots, m$. Hence, an active constraint corresponds to $|kx_i + l - y_i| = z$. If $z = 0$, all constraints in (LP) are active. If $z > 0$, for each i, at most one of the constraints $-z \leq$

$kx_i + l - y_i$ and $kx_i + l - y_i \le z$ an be active. Hence, an optimal extreme point has at least three active constraints corresponding to three different indices, which means at least three different indices for which $|kx_i + l - y_i| = z$, i.e., at least three points at which $|kx_i + l - y_i| = z$.

In the above, we have implicitly assumed that (LP) is three-dimensional, which corresponds to the constraint matrix in (DLP) having full row rank. To be precise, we should also show that this is the case, so that the standard analysis applies. This is more of a technicality. To see that the constraint matrix of (DLP) has full row rank, assume that there is a linear combination of the rows of the constraint matrix which gives the zero vector, i.e., there are α, β and γ such that

$$\begin{aligned} x_i\alpha + \beta + \gamma = 0, \quad & i = 1, \ldots, n, \\ -x_i\alpha - \beta + \gamma = 0, \quad & i = 1, \ldots, n. \end{aligned}$$

We now need to show that $\alpha = \beta = \gamma = 0$. Adding the two equations for a given i gives $\gamma = 0$. Taking two different indices i and j gives $(x_i - x_j)\alpha = 0$. Consequently, $\alpha = 0$, since $x_i \ne x_j$ by the statement. Thus, $\beta = 0$, and we conclude that the constraint matrix has full row rank.

We can now make the statement precise. Since (LP) is feasible with bounded optimal value, it follows by strong duality that (DLP) is feasible with the same optimal value. Hence, if we solve (DLP) by the simplex method, we obtain a final basic feasible solution with a basis matrix of dimension 3×3. Corresponding to this matrix, there are three constraints in the primal that are satisfied with equality. The above argument thus applies.

5. (a) For a fix vector $u \in I\!R^n$, Lagrangian relaxation of the first set of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n \left(-u_i + \sum_{j=1}^n (u_i - c_{ij}) x_{ij} \right) + \sum_{j=1}^n f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \le b_j z_j, \quad j = 1, \ldots, n, \\ & x_{ij} \in \{0,1\}, \quad i = 1, \ldots, n, \ j = 1, \ldots, n, \\ & z_j \in \{0,1\}, \quad j = 1, \ldots, n, \end{array}$$

where a_i, $i = 1, \ldots, n$, b_j, $j = 1, \ldots, n$, f_j, $j = 1, \ldots, n$, and c_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n$, are nonnegative integer constants.

(b) For a fix vector $v \in I\!R^n$, Lagrangian relaxation of the second group of constraints gives

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n \sum_{j=1}^n (a_i v_j - c_{ij}) x_{ij} + \sum_{j=1}^n (f_j - b_j v_j) z_j \\ \text{subject to} & \displaystyle\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \ldots, n, \\ & x_{ij} \in \{0,1\}, \quad i = 1, \ldots, n, \ j = 1, \ldots, n, \\ & z_j \in \{0,1\}, \quad j = 1, \ldots, n, \end{array}$$

where a_i, $i = 1, \ldots, n$, b_j, $j = 1, \ldots, n$, f_j, $j = 1, \ldots, n$, and c_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n$, are nonnegative integer constants.

(c) The first relaxation decomposes into one separate problem for each j according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^n (u_i - c_{ij}) x_{ij} + f_j z_j \\ \text{subject to} & \displaystyle\sum_{i=1}^n a_i x_{ij} \le b_j z_j, \\ & x_{ij} \in \{0,1\}, \quad i = 1, \ldots, n, \\ & z_j \in \{0,1\}, \end{array}$$

for $j = 1, \ldots, n$. We can here solve two problems, for $z_j = 0$ and $z_j = 1$, and then take the minimum. For $z_j = 0$, the solution is given by $x_{ij} = 0$, $j = 1, \ldots, n$. For $z_j = 1$, we obtain a binary knapsack problem, which may for example be solved using dynamical programming.

The second relaxation decomposes into trivial problems. For the z-variables we obtain for each i according to

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^n (f_j - b_j v_j) z_j \\ \text{subject to} & z_j \in \{0,1\}, \quad j = 1, \ldots, n, \end{array}$$

which can be solved directly with $z_j = 1$ if $f_j - b_j v_j < 0$ and $z_j = 0$ if $f_j - b_j v_j \ge 0$ for $j = 1, \ldots, n$. For the x-variables we obtain

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{j=1}^n (a_i v_j - c_{ij}) x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^n x_{ij} = 1, \\ & x_{ij} \in \{0,1\}, \quad j = 1, \ldots, n, \end{array}$$

for $i = 1, \ldots, n$. These can be solved directly by noting which x_{ij}-variable having the smallest coefficient in the objective function.

(d) The second relaxation gives a relaxed problem which gives integer optimal solutions even if one relaxes the integer constraint. Hence, the corresponding dual underestimation becomes identical with the one obtained if performing an LP-relaxation.

The first relaxation gives a more complicated relaxed problem, and here one can expect the underestimation to be better than one would obtain with an LP-relaxation.