SF2812 Applied linear optimization, final exam
Monday March 12 2018 8.00–13.00
Brief solutions

1. (a) The primal variables x are the dual variables of the constraints $A^T y \le c$ in (DLP). Therefore, the marginal costs for "`EQU cons`" give $x = (2\ 1\ 0\ 0\ 1\ 3)^T$.

(b) We assume that the given solution is a basic feasible solution, i.e., that columns 1, 2, 5 and 6 of A form a nonsingular basis matrix B. As long as the change in c_1 gives the same optimal basis, we expect the change in optimal solution to be given by $c_B^T x_B + \delta e_1^T x$, i.e., $5 + 2\delta$.

(c) We have $B^T y = c_B$. If c_B is changed to $c_B + \delta e_1$, we get the corresponding dual solution y_δ by $B^T y_\delta = c_B + \delta e_1$, i.e., $y_\delta = y + \delta\eta$, where $B^T \eta = e_1$. Insertion of numerical values gives

$$\begin{pmatrix} 1 & 2 & 1 & -1 \\ 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The assumption that B is nonsingular is valid, and the solution is given by $\eta = (-1\ 1\ 0\ 0)^T$. The bound on δ is then given by dual feasibility, i.e., $N^T(y + \delta\eta) \le c_N$ or equivalently $\delta N^T \eta \le c_N - N^T y$. Insertion of numerical values gives

$$\delta \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \le \begin{pmatrix} 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} 0 &\le 1, \\ -2\delta &\le 2, \end{aligned}$$

The bound is consequently given by $\delta \ge -1$. Therefore, the optimal value is given by $5 + 2\delta$ for $\delta \ge -1$.

2. (a) The system of primal-dual nonlinear equations is given by

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 2, && (1a) \\ y + s_1 &= 1, && (1b) \\ y + s_2 &= 1, && (1c) \\ 2y + s_3 &= 3, && (1d) \\ x_1 s_1 &= \mu, && (1e) \\ x_2 s_2 &= \mu, && (1f) \\ x_3 s_3 &= \mu. && (1g) \end{aligned}$$

where we also implicitly require $x > 0$ and $s > 0$.

We will throughout use the fact that the problem is small and has a particular structure. We may use (1b)–(1g) to express x and s as a function of y according to

$$s_1 = s_2 = 1-y, \quad s_3 = 3-2y, \quad x_1 = x_2 = \frac{\mu}{s_1} = \frac{\mu}{1-y}, \quad x_3 = \frac{\mu}{s_3} = \frac{\mu}{3-2y}.$$

Insertion of the expressions for x into (1a) gives

$$\frac{\mu}{1-y} + \frac{\mu}{3-2y} = 1.$$

(b) For completeness, we derive the expression for $y(\mu)$. This is not asked for in the question. If both sides of the equation in y are multiplied by $(1-y)(3-2y)$, we obtain

$$(3-2y)\mu + (1-y)\mu = (1-y)(3-2y),$$

which can be simplified to

$$y^2 - \frac{5-3\mu}{2}y + \frac{3-4\mu}{2} = 0,$$

Solving this equation gives

$$y = \frac{5-3\mu}{4} - \sqrt{\frac{(5-3\mu)^2}{16} - \frac{3-4\mu}{2}} = \frac{5-3\mu}{4} - \frac{\sqrt{1+2\mu+9\mu^2}}{4},$$

where the minus sign has been chosen to make $y < 1$, required by $s_1 = 1-y > 0$. Taylor series expansion of $\sqrt{1+2\mu+9\mu^2}$ gives

$$\sqrt{1+2\mu+9\mu^2} = 1 + \frac{2\mu+9\mu^2}{2} - \frac{(2\mu+9\mu^2)^2}{8} + o(\mu^2) = 1+\mu+4\mu^2+o(\mu^2).$$

Insertion into $y(\mu)$ gives

$$\begin{aligned} y(\mu) &= \frac{5-3\mu}{4} - \frac{\sqrt{1+2\mu+9\mu^2}}{4} = \frac{5-3\mu}{4} - \frac{1+\mu+4\mu^2}{4} + o(\mu^2) \\ &= 1-\mu-\mu^2+o(\mu^2), \end{aligned}$$

which is the given expression.
The answer to the question starts here. The given expression for $y(\mu)$ gives

$$\begin{aligned} s_1(\mu) &= s_2(\mu) = 1 - y(\mu) \approx 1-(1-\mu) = \mu, \\ s_3(\mu) &= 3-2y(\mu) \approx 3-2(1-\mu) = 1+2\mu, \\ x_1(\mu) &= x_2(\mu) = \frac{\mu}{1-y(\mu)} \approx \frac{\mu}{1-1+\mu+\mu^2} = \frac{1}{1+\mu} \approx 1-\mu, \\ x_3(\mu) &= \frac{\mu}{3-2y(\mu)} = \frac{\mu}{3-2(1-\mu)} = \frac{\mu}{1+2\mu} \approx \mu, \end{aligned}$$

where in all cases $o(\mu)$ approximations have been derived.

(c) Letting $\mu \to 0$ gives

$$x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad y = 1, \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is straightforward to verify that $Ax = b$, $x \geq 0$, $A^T y + s = c$, $s \geq 0$, $x^T s = 0$. Consequently, optimality holds. The given x is not a basic feasible solution, which can be seen from two positive components of x with only one row in A. This situation arises because this particular primal problem does not have a unique optimal solution. The barrier trajectory leads to a basic feasible solution only if the primal problem has a unique solution.

3. The suggested initial extreme points $v_1 = (-1\ \ 0\ \ 1\ \ 0)^T$ and $v_2 = (-1\ \ 0\ \ 1\ \ 0)^T$ give the initial basis matrix

$$B = \begin{pmatrix} Av_1 & Av_2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}.$$

The right-hand side in the master problem is $b = (-2\ 1)^T$. Hence, the basic variables are given by

$$\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1\ c^T v_2) = (-4\ \ -2)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

By forming $c^T - y_1 A = (1\ \ 0\ \ 0\ \ 4)$ we obtain the subproblem

$$\begin{array}{lll} 1- & \text{minimize} & x_1 + 4x_4 \\ & \text{subject to} & -1 \leq x_1 + x_2 \leq 1, \\ & & -1 \leq x_1 - x_2 \leq 1, \\ & & -1 \leq x_3 + x_4 \leq 1, \\ & & -1 \leq x_3 - x_4 \leq 1. \end{array}$$

An optimal extreme point to the subproblem is given by $v_3 = (-1\ \ 0\ \ 0\ \ -1)^T$ with optimal value -4. Hence, α_3 should enter the basis. The corresponding column in the master problem is given by

$$\begin{pmatrix} Av_3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The change to the basic variables is given by

$$\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Finding the maximum step η for which $\alpha + \eta p \geq 0$ gives

$$\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \eta \begin{pmatrix} 0 \\ -1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e., $\eta = 1/2$ so that α_2 leaves the basis.

Hence, the new basis corresponds to v_1 and v_3 so that

$$B = \begin{pmatrix} Av_1 & Av_3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}.$$

The basic variables are given by

$$\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} \alpha_1 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

The cost of the basic variables are given by $(c^T v_1 \; c^T v_3) = (-4 \; -6)$. Consequently, the simplex multipliers are given by

$$\begin{pmatrix} -3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -7 \end{pmatrix}.$$

By forming $c^T - y_1 A = (5 \;\; 2 \;\; -2 \;\; 2)$ we obtain the subproblem

$$\begin{array}{lll} 7+ & \text{minimize} & 5x_1 + 2x_2 - 2x_3 + 2x_4 \\ & \text{subject to} & -1 \le x_1 + x_2 \le 1, \\ & & -1 \le x_1 - x_2 \le 1, \\ & & -1 \le x_3 + x_4 \le 1, \\ & & -1 \le x_3 - x_4 \le 1. \end{array}$$

Both v_1 and v_3 are optimal extreme points to the subproblem, so the optimal value of the subproblem is 0. Hence, the master problem has been solved. The solution to the original problem is given by

$$v_1\alpha_1 + v_3\alpha_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} -1 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

The optimal value is –5.

4. (See the course material.)

5. (a) We have

$$\phi_x(\lambda) = \lambda + \frac{1}{\alpha} \sum_{i:d_i > \lambda} \Delta_i (d_i - \lambda),$$

since $(d_i - \lambda)_+ = 0$ for $d_i \le \lambda$. If $\lambda \ne d_j$, $j = 1, \dots, n$, then $\{i : d_i > \lambda\}$ is constant in a neighborhood of λ. Therefore,

$$\frac{d\phi_x(\lambda)}{d\lambda} = 1 - \frac{1}{\alpha} \sum_{i:d_i > \lambda} \Delta_i.$$

We first consider $\lambda < d_{m_\alpha}$. Then, by the ordering of the d_is, $d_i > \lambda$, $i = 1, \ldots, m_\alpha$, so that

$$\frac{d\phi_x(\lambda)}{d\lambda} \le 1 - \frac{1}{\alpha}\sum_{i=1}^{m_\alpha} \Delta_i = \frac{1}{\alpha}\left(\alpha - \sum_{i=1}^{m_\alpha} \Delta_i\right) < 0,$$

if $\lambda \ne d_j$, $j = 1, \ldots, m_\alpha - 1$, where the last inequality follows from the definition of m_α.

Analogously, if $\lambda > d_{m_\alpha}$, then $d_i < \lambda$, $i = m_\alpha, m_\alpha + 1, \ldots, m$. Therefore,

$$\frac{d\phi_x(\lambda)}{d\lambda} \ge 1 - \frac{1}{\alpha}\sum_{i=1}^{m_\alpha - 1} \Delta_i = \frac{1}{\alpha}\left(\alpha - \sum_{i=1}^{m_\alpha - 1} \Delta_i\right) \ge 0,$$

if $\lambda \ne d_j$, $j = m_\alpha + 1, \ldots, n$, where the last inequality again follows from the definition of m_α.

Consequently, ϕ_x is a piecewise linear function which is decreasing for $\lambda < d_{m_\alpha}$ and nondecreasing for $\lambda \ge d_{m_\alpha}$. We conclude that d_{m_α} is a global minimizer of ϕ_x.

(b) We may introduce a new variable u and rewrite (P_α) as

$$\begin{array}{ll} \text{minimize} & u \\ \text{subject to} & u \ge \lambda + \dfrac{1}{\alpha}\displaystyle\sum_{i=1}^{m} \Delta_i (d_i - \lambda)_+, \\ & d = Px, \\ & x \ge 0. \end{array}$$

It remains to rewrite $(d_i - \lambda)_+$. Since $(d_i - \lambda)_+$ is to be minimized, we may introduce a new variable μ_i and require $\mu_i \ge d_i - \lambda$ and $\mu_i \ge 0$. Then, the minimum value of μ_i equals $(d_i - \lambda)_+$.

The resulting linear program takes the form

$$\begin{array}{ll} \text{minimize} & u \\ \text{subject to} & u \ge \lambda + \dfrac{1}{\alpha}\displaystyle\sum_{i=1}^{m} \Delta_i \mu_i, \\ & \mu_i \ge d_i - \lambda, \quad i = 1, \ldots, m, \\ & \mu_i \ge 0, \quad i = 1, \ldots, m, \\ & d = Px, \\ & x \ge 0. \end{array}$$

Remark: For an example when CVaR is used in radiation therapy optimization, see L. Engberg, A. Forsgren, K. Eriksson and B. Hårdemark, *Explicit optimization of plan quality measures in intensity-modulated radiation therapy treatment planning*. Medical Physics 44 (2017) 2045-2053.

A challenge in this application is that $m \gg n$. In the paper, a tailored interior point method for handling $m \gg n$ is designed to solve the linear program.