KTH Mathematics

# SF2812 Applied linear optimization, final exam
## Tuesday October 23 2007 14.00–19.00

*Examiner:* Anders Forsgren, tel. 790 71 27.

*Allowed tools:* Pen/pencil, ruler and rubber; plus a calculator provided by the department.

*Solution methods:* Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

*Note!* Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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**1.** Let $(LP)$ be defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 3 & 4 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 1 & 3 & 2 & 0 & 0 \end{pmatrix}^T.$$

(a) A person named AF has used GAMS to model and solve this problem. The GAMS input file can be found at the end of the exam. Unfortunately, AF has lost the GAMS output file. He does have a partial GAMS output file, which reads:

```
---- EQU cons

      LOWER      LEVEL      UPPER     MARGINAL

i1     4.000      4.000      4.000      0.333
i2     3.000      3.000      3.000       EPS
i3     4.000      4.000      4.000       EPS

                                 LOWER      LEVEL      UPPER     MARGINAL

---- EQU objfun                    .          .          .        -1.000
```

```
                              LOWER      LEVEL      UPPER    MARGINAL

---- VAR obj                   -INF      1.333      +INF        .

---- VAR x

      LOWER     LEVEL     UPPER    MARGINAL

j1      .        1.333     +INF        .
j2      .          .       +INF      1.667
j3      .          .       +INF      1.667
j4      .        4.333     +INF        .
j5      .        1.333     +INF        .
```

AF has run several versions of the problem, and he is not sure that this file is the one that corresponds to $(LP)$. Help AF by showing that this file gives the optimal solution to $(LP)$. ............................................(4p)

(b) The reason that AF has run several versions of the file is that the coefficients for $x_2$ and $x_3$ in the objective function are not very precise. Hence, in addition to wanting an optimal solution to $(LP)$, he also wants to know how sensitive the optimal solution is to changes in $c_2$ and $c_3$. He knows that the fluctuations in $c_2$ and $c_3$ are at most one unit up and down. AF is not an optimization expert, and he has considered asking an expert about assistance in setting up a stochastic programming model. Give AF a qualified advice on what to do. ..................................................................(4p)

(c) AF is also worried about $b_1$. He has been asked how sensitive the optimal value is to changes in $b_1$. Help AF to provide this information. ................(2p)

**2.** Consider the linear programming problem $(LP)$ and its dual $(DLP)$ defined as

$$(LP)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array} \qquad (DLP)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \ge 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 4 & 5 & -1 & 3 & 2 \\ -1 & 3 & 2 & 0 & 4 & 0 \\ -1 & 2 & 4 & 3 & 0 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 26 \\ 15 \\ 21 \end{pmatrix},$$

$$c = \begin{pmatrix} 1 & 1.9 & 3.85 & 2 & 0 & 3 \end{pmatrix}^T.$$

The related barrier transformed problem $(P_\mu)$, defined by

$$(P_\mu)\quad \begin{array}{ll} \text{minimize} & c^T x - \mu \sum_{j=1}^{6} \ln x_j \\ \text{subject to} & Ax = b, \\ & (x > 0), \end{array}$$

has an optimal solution $x(\mu)$ and lagrange multiplier vector $y(\mu)$ for $\mu = 10^{-3}$ which numerically is given by approximately

```
xmu =
   0.000769682525151
   2.961435517128881
   3.018504931147981
   1.000626320521990
   0.019863317210637
   0.000999981054749

ymu =
   0.250242738914173
  -0.200268068764611
   0.749747788280888
```

(a) Use the above numbers to give an approximate solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the primal-dual nonlinear equations, associated with a primal-dual interior method for solving $(LP)$, for $\mu = 10^{-3}$. .................................. (4p)

(b) The above problem $(LP)$ has an optimal solution which is integer valued, and there is an optimal solution to $(DLP)$ for which the components of $y$ are integer multiples of 1/4 or 1/5. Given this knowledge, use your results from exercise 2a to make a qualified guess of optimal solutions to $(LP)$ and $(DLP)$ respectively. Motivate your guess and verify optimality. ............................. (4p)

(c) If the simplex method had been used to solve $(LP)$, would the same primal optimal solution have been obtained? Comment on the result. .......... (2p)

*Hint:* It holds that

$$\det \begin{pmatrix} 4 & 5 & -1 \\ 3 & 2 & 0 \\ 2 & 4 & 3 \end{pmatrix} \neq 0.$$

**3.** Consider the stochastic program $(P)$ given by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where $\omega$ is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that $\omega$ takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let $T_i$ denote $T(\omega_i)$ and let $h_i$ denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \dots, N, \\ & x \ge 0, \\ & y_i \ge 0, \quad i = 1, \dots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., $W$ does not depend on $i$.) ............ (6p)

(b) Define *VSS* in terms of suitable optimization problems. ............ (2p)

(c) Define *EVPI* in terms of suitable optimization problems. ............ (2p)

**4.** Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - \sum_{j=1}^{n} f_j z_j \\ \text{subject to} & \sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \dots, n, \\ & \sum_{i=1}^{n} a_i x_{ij} \ge b_j z_j, \quad j = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \ z_j \in \{0, 1\}, \quad i = 1, \dots, n, \ j = 1, \dots, n, \end{array}$$

where $a_i$, $i = 1, \dots, n$, $b_j$, $j = 1, \dots, n$, $c_{ij}$, $i = 1, \dots, n$, $j = 1, \dots, n$, and $f_j$, $j = 1, \dots, n$, are integer nonnegative constants.

(a) Formulate the Lagrangian relaxed problem arising from relaxing the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \dots, n.$$

Simplify the formulation as much as you can. ............ (4p)

(b) Formulate the Lagrangian relaxed problem arising from relaxing the constraints

$$\sum_{i=1}^{n} a_i x_{ij} \ge b_j z_j, \quad j = 1, \dots, n.$$

Simplify the formulation as much as you can. ............ (4p)

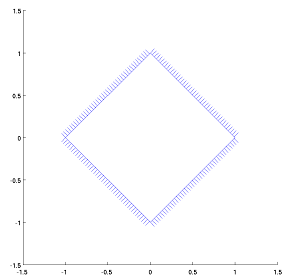
(c) Discuss which of the two dual problems, associated with each of the two relaxations, that should give the best underestimate of the optimal value of the original problem. ............ (2p)

**5.** Consider the linear program

$$
(LP) \quad \begin{array}{ll}
\text{minimize} & 7x_1 + 6x_2 + 5x_3 + 3x_4 \\
\text{subject to} & 3x_1 + 2x_2 + 4x_3 + 5x_4 = 6, \\
& -1 \le x_1 + x_2 \le 1, \\
& -1 \le x_1 - x_2 \le 1, \\
& -1 \le x_3 + x_4 \le 1, \\
& -1 \le x_3 - x_4 \le 1.
\end{array}
$$

Solve $(LP)$ by Dantzig-Wolfe decomposition. Consider $3x_1+2x_2+4x_3+5x_4 = 6$ the complicating constraint. Start with the initial basis corresponding to the extreme points $(1\ 0\ 0\ 1)^T$ and $(-1\ 0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. ............................................(10p)

*Hint 1:* The following figure may be helpful:




*Hint 2:* The following relation for inversion of a two-by-two matrix may be useful for hand calculation:

$$
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \quad \text{if} \quad \alpha\delta - \beta\gamma \neq 0.
$$

*Good luck!*

GAMS file for exercise 1:

```
set rows      / i1*i3 /;

set cols      / j1*j5 /;

table A(rows,cols)
         j1   j2   j3   j4   j5
i1        3    4    1    0    0
i2       -1    2    0    1    0
i3        2    1    0    0    1 ;

parameter c(cols)
          / j1 1
            j2 3
            j3 2 /;

parameter b(rows)
          / i1 4
            i2 3
            i3 4 /;

variables
        obj
        x(cols);

positive variables x;

equations
        cons(rows)
        objfun;

objfun .. sum(cols,c(cols)*x(cols)) - obj =e= 0;
cons(rows) .. sum(cols,A(rows,cols)*x(cols)) =e= b(rows);

model LPex / all /;

solve LPex using lp minimizing obj;
```