KTH Mathematics

SF2812 Applied linear optimization, final exam
Wednesday January 16 2008 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and rubber; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (LP) be defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 4 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 3 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 8 & 7 & -1 & -1 & 0 \end{pmatrix}^T.$$

Assume that we want to solve (LP) by a primal-dual interior method.

Let the initial point be given by

$$x = \begin{pmatrix} 1 \\ 2 \\ 8 \\ 2 \\ 4 \end{pmatrix}, \quad y = \begin{pmatrix} 1.1 \\ 1.1 \\ 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1.7 \\ 0.5 \\ 0.1 \\ 0.1 \\ 1 \end{pmatrix}.$$

(a) Formulate the system of linear equations to be solved at the initial iteration of a primal-dual interior method, starting from the given values. Set up the equations algebraically, and then insert numerical values. Assign suitable values to parameters that you may require. .... (7p)
Note: Please do *not* solve the linear system of equations.

(b) Assume that you have computed a solution Δx, Δy, Δs from the system of linear equations formulated in Exercise 1a. Show how you would generate the next iterate. .... (3p)

Note: Please do *not* solve the linear system of equations.

2. Let (LP) be defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 4 & -1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 3 & 1 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 8 & 7 & -1 & -1 & 0 \end{pmatrix}^T.$$

A person named AF claims that he has obtained $x^* = (0\ 1\ 2\ 0\ 0)^T$ as an optimal solution to (LP) by the simplex method. However, he is a bit confused, since he would expect an optimal solution to have three positive components. Your task is to clarify AF's confusion.

(a) Show that x^* is a basic feasible solution to (LP). .... (2p)

(b) AF is probably confused since x^* is a degenerate basic feasible solution. Solve (LP) by the simplex method three times, starting from each of the feasible bases given by (i) x_1, x_2, x_3, (ii) x_2, x_3, x_4, (ii) x_2, x_3, x_5. Was AF's claim correct? Comment on your results. .... (8p)

3. Consider the binary integer programming problem (IP) given by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -5x_1 - 7x_2 - 10x_3 \\ \text{subject to} & -3x_1 - 6x_2 - 7x_3 \geq -8, \\ & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & x_j \in \{0, 1\}, \quad j = 1, \dots, n. \end{array}$$

Assume that the constraint $-3x_1 - 6x_2 - 7x_3 \geq -8$ is relaxed by Lagrangian relaxation for a nonnegative multiplier u.

(a) For $u = 1$, compute two optimal solutions to the resulting Lagrangian relaxed problem. The Lagrangian relaxed problem may be solved by any method, that need not be systematic. .... (4p)

(b) Use the two optimal solutions to the Lagrangian relaxed problem computed in Exercise 3a to give two different subgradients to the dual objective function φ at $u = 1$. .... (4p)

(c) Show that there is a convex combination of the two subgradients computed in Exeercise 3b that gives the zero vector. What is the implication for the dual problem? . . . . (2p)

4. Consider the integer program (IP) defined by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0, \quad x \text{ integral.} \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D) \qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \geq 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \geq d, x \geq 0 \text{ integral}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integrality requirement, i.e.,

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0. \end{array}$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. . . . . (10p)

5. Consider the linear program (LP) given by

$$(LP) \qquad \begin{array}{ll} \text{minimize} & x_1 - x_2 + x_3 + x_4 \\ \text{subject to} & -x_1 + 2x_2 + x_3 - 3x_4 = 1, \\ & -1 \leq x_j \leq 1, \quad j = 1, \dots, 4. \end{array}$$

Solve (LP) by Dantzig-Wolfe decomposition. Consider $-x_1 + 2x_2 + x_3 - 3x_4 = 1$ the complicating constraint. Start with the initial basis corresponding to the extreme points $(-1 \;\; 1 \;\; -1 \;\; 1)^T$ and $(-1 \;\; 1 \;\; 1 \;\; -1)^T$. The subproblems that arise may be solved by any method, that need not be systematic. . . . . (10p)

Good luck!