KTH Mathematics

SF2812 Applied linear optimization, final exam
Monday October 20 2008 8.00–13.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser; plus a calculator provided by the department.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a transportation problem (TP) defined as

$$(TP)\qquad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1,2,3, \\ & \displaystyle\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3,4, \\ & x_{ij} \ge 0, \quad i = 1,2,3, \ j = 1,2,3,4, \end{array}$$

where

$$C = \begin{pmatrix} 4 & 2 & 5 & 1 \\ 7 & 4 & 7 & 5 \\ 7 & 5 & 6 & 2 \end{pmatrix}, \quad a = \begin{pmatrix} 10 \\ 12 \\ 10 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 8 \\ 7 \\ 9 \end{pmatrix}.$$

The dual problem associated with (TP) may be written as

$$(DTP)\qquad \begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{3} a_i u_i + \sum_{j=1}^{4} b_j v_j \\ \text{subject to} & u_i + v_j \le c_{ij}, \quad i = 1,2,3, \quad j = 1,2,3,4. \end{array}$$

You have been given $\widehat{X}$, $\widehat{u}$ and $\widehat{v}$ as

$$\widehat{X} = \begin{pmatrix} 8 & 1.5 & 0 & 0.5 \\ 0 & 6.5 & 5.5 & 0 \\ 0 & 0 & 1.5 & 8.5 \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 0 \end{pmatrix}.$$

(a) A friend of yours, who has not taken this course, claims that $\widehat{X}$ cannot be optimal to (TP), since the transportation problem should have integer valued optimal solutions when a and b are integers. Comment on your friend's claim. .... (2p)

(b) Verify that $\widehat{X}$ is optimal to (TP) and that $\widehat{u}$, $\widehat{v}$ is optimal to (DTP). ... (3p)

(c) Find, using $\widehat{X}$, two integer valued optimal solutions to (TP). .... (3p)
Hint: It holds that $\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}u_{ij} = 0$, $\sum_{j=1}^{4} u_{ij} = 0$, $i = 1,2,3$, and $\sum_{i=1}^{3} u_{ij} = 0$, $j = 1,2,3,4$, for

$$U = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(d) Explain why you would not obtain $\widehat{X}$ as an answer if you used the simplex method to solve (TP). .... (2p)

2. Consider the linear program (LP) defined by

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where A is a given $m \times n$-matrix with linearly independent rows. Let $S = \{x : Ax = b, x \geq 0\}$.

(a) Define a convex set. .... (2p)

(b) Show that S is a convex set. .... (2p)

(c) Define a basic feasible solution to (LP). .... (3p)

(d) Show that x is an extreme point to S if and only if x is a basic feasible solution to (LP). .... (3p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & -1 & 2 & 1 \end{pmatrix}^T.$$

Assume that we want to solve (LP) using a primal-dual interior-point method. Assume further that we initially choose $x^{(0)} = (2\,2\,2\,1)^T$, $y^{(0)} = (0\,0)^T$, $s^{(0)} = (3\,1\,1\,1)^T$, and $\mu = 1$. Here, y and s denote the dual variables.

(a) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior-point method for the given initial values. First formulate the general form and then add explicit numerical values into the system of equations. .... (7p)

(b) The solution to the above system of linear equations is given by

$$\Delta x = \begin{pmatrix} -0.6 \\ 1 \\ -4.2 \\ -0.2 \end{pmatrix}, \quad \Delta y = \begin{pmatrix} -1.8 \\ 0.4 \end{pmatrix}, \quad \Delta s = \begin{pmatrix} -1.6 \\ -1 \\ 1.6 \\ 0.2 \end{pmatrix}.$$

Use these values to determine $x^{(1)}$, $y^{(1)}$ and $s^{(1)}$ in a suitable way. .... (3p)

4. Consider fitting a line $y = kx + l$ to a number of given points (x_i, y_i), $i = 1, \dots, m$. In particular, k and l should be chosen so that the maximum deviation in the y-direction is minimized, i.e., k and l are chosen according to $\min_{k,l} \max_{i=1,\dots,m} |kx_i + l - y_i|$. By introducing the extra variable z, the problem may be written as an LP problem on the form

$$(LP) \quad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & -z \le kx_i + l - y_i \le z, \quad i = 1, \dots, m, \end{array}$$

where x_i, $i = 1, \dots, m$ and y_i, $i = 1, \dots, m$, are given parameters, and k, l and z are the variables. We assume that $m \ge 3$ and $x_i \ne x_j$ for $i \ne j$.

(a) Form the dual problem (DLP) associated with (LP). .... (5p)

(b) Given an optimal solution k, l, z to this line fitting problem, show that there are at least three points among the given points (x_i, y_i), $i = 1, \dots, m$, at which $|kx_i + l - y_i| = z$. You should motivate this result based on properties of the linear programs. .... (5p)

5. Consider the integer linear programming problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} + \sum_{j=1}^n f_j z_j \\ \text{subject to} & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \\ & \sum_{i=1}^n a_i x_{ij} \le b_j z_j, \quad j = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \ j = 1, \dots, n, \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

where a_i, $i = 1, \dots, n$, b_j, $j = 1, \dots, n$, f_j, $j = 1, \dots, n$, and c_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$, are nonnegative integer constants.

(a) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, n,$$

are relaxed by Lagrangian relaxation. .... (2p)

(b) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{i=1}^{n} a_i x_{ij} \leq b_j z_j, \quad j = 1, \ldots, n,$$

are relaxed by Lagrangian relaxation. .... (2p)

(c) Describe how the Lagrangian relaxed problems can be solved in the two cases. .... (4p)

(d) Discuss which of the two Lagrangian relaxations that would give the best under-estimate of the optimal value of the original problem when the corresponding dual problem is solved. .... (2p)

Good luck!