KTH Mathematics

SF2812 Applied linear optimization, final exam
Monday October 19 2009 14.00–19.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (P) and (D) be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \ge 0. \end{array}$$

For a fixed positive barrier parameter μ, consider the primal-dual nonlinear equations

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

where we in addition require $x > 0$ and $s > 0$. Here, $X = \mathrm{diag}(x)$, $S = \mathrm{diag}(s)$ and e is an n-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive μ, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to (P) and $y(\mu), s(\mu)$ are feasible to (D) with duality gap $n\mu$. . . . . (3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton's method. . . . . (5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of μ? . . . . (2p)

2. Consider a mixed-integer linear programming problem with one integer variable (but a large number of continuous variables). Assume that this problem is solved by branch-and-bound with linear programming relaxation at the nodes. Show that the branch-and-bound tree will have at most three nodes. You may assume that the linear programs that arise have unique optimal solutions. .... (10p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & 3 & 2 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} -3 & 3 & 2 & 0 \end{pmatrix}^T.$$

An optimal basic feasible solution has been computed for $b_2 = -1$. This solution is $\widetilde{x} = (1\ 0\ 0\ 1)^T$. The corresponding dual optimal solution is $\widetilde{y} = (-1\ 1)^T$ and $\widetilde{s} = (0\ 1\ 1\ 0)^T$.

Unfortunately, the value of b_2 was not correct. The correct value is $b_2 = 3$. Now, $\widetilde{x}$ is not feasible to the correct primal problem, whereas $\widetilde{y}$ and $\widetilde{s}$ are feasible to the correct dual problem. Solve the correct problem by the dual simplex method, starting from $\widetilde{y}$ and $\widetilde{s}$. .... (10p)

4. Consider the linear programming problem (LP) given by

$$(LP) \qquad \begin{array}{ll} \text{minimize} & 3x_1 - 2x_2 + 4x_3 - x_4 \\ \text{subject to} & -2x_1 - x_2 - 4x_3 + x_4 = 1, \\ & -2 \leq 2x_1 - x_2 \leq 2, \\ & -2 \leq 2x_1 + x_2 \leq 2, \\ & -2 \leq 2x_3 - x_4 \leq 2, \\ & -2 \leq 2x_3 + x_4 \leq 2. \end{array}$$

Your task is to solve (LP) using Dantzig-Wolfe decomposition. Consider the equality constraint $-2x_1 - x_2 - 4x_3 + x_4 = 1$ as the hard constraint. For

$$S = \left\{ x \in I\!R^4 : -2 \leq 2x_j - x_{j+1} \leq 2,\ -2 \leq 2x_j + x_{j+1} \leq 2,\ j = 1, 3 \right\},$$

write $x \in S$ as a convex combination of the extreme points of S. In the master problem, start with the basis that corresponds to the extreme points $(0\ \ 2\ \ -1\ \ 0)^T$ and $(0\ \ 2\ \ 0\ \ 2)^T$. The subproblem(s) that arise(s) may be solved in a nonsystematic way, e.g., graphically. .... (10p)

5. Consider an integer programming problem posed as a transportation problem with a time constraint in the form

$$
(IP) \qquad
\begin{array}{lll}
\text{minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = a_i, & i = 1,\ldots,m, \\
& \displaystyle\sum_{i=1}^{m} x_{ij} = b_j, & j = 1,\ldots,n, \\
& \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} \le T, & \\
& x_{ij} \in \{0,1\}, & i = 1,\ldots,m,\ j = 1,\ldots,n,
\end{array}
$$

where c_{ij}, a_i, b_j, t_{ij} and T are positive constants for $i = 1,\ldots,m$, $j = 1,\ldots,n$.

(a) Assume that the constraint

$$\sum_{i=1}^{m}\sum_{j=1}^{n} t_{ij}x_{ij} \le T$$

is relaxed by Lagrangian relaxation. For a given Lagrange multiplier u, with $u \ge 0$, formulate the relaxed problem. Let $x(u)$ denote an optimal solution to this Lagrangian relaxed problem. Give an expression for a subgradient to the corresponding dual objective function at u. .... (4p)

(b) Assume that the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1,\ldots,m, \quad \sum_{i=1}^{m} x_{ij} = b_j,\ j = 1,\ldots,n,$$

are relaxed by Lagrangian relaxation. For given Lagrange multipliers v_i, $i = 1,\ldots,m$, and w_j, $j = 1,\ldots,n$, respectively, formulate the relaxed problem. Let $x(v,w)$ denote an optimal solution to this Lagrangian relaxed problem. Give an expression for a subgradient to the corresponding dual objective function at (v,w). .... (4p)

(c) Each of the two relaxations above gives rise to a corresponding dual problem. Which of these dual problems would you expect to give the tightest underestimate of the optimal value of (IP)? .... (2p)

Good luck!