KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday October 21 2010 14.00–19.00

Examiner: Anders Forsgren, tel. 790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (P) and (D) be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

For a fixed positive barrier parameter μ, consider the primal-dual nonlinear equations

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

where we in addition require $x > 0$ and $s > 0$. Here, $X = \operatorname{diag}(x)$, $S = \operatorname{diag}(s)$ and e is an n-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive μ, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to (P) and $y(\mu), s(\mu)$ are feasible to (D) with duality gap $n\mu$. . . . . (3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton's method. . . . . (5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of μ? . . . . (2p)

2. Consider the linear programming problem (LP) defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 16 \\ 16 \end{pmatrix},$$
$$c = \begin{pmatrix} -1 & 1 & 1 & 0 & 0 \end{pmatrix}^T.$$

A friend of yours claims that she has computed an optimal solution $\widehat{x} = (3\ 5\ 2\ 0\ 2)^T$ by an interior method. However, she has then been asked to provide two optimal basic feasible solutions.

Help your friend by providing two basic feasible solutions with the same objective function value as $\widehat{x}$. Start from $\widehat{x}$. Finally, verify optimality of one of these basic feasible solutions. .... (10p)

Hint: Your may find one or several of the results below useful.

$$\begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 0 \\ -4 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 3 & 1 & -1 & 0 & 0 \\ 2 & 2 & 0 & -1 & 0 \\ 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ -8 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

3. Consider the primal-dual pairs of linear programs defined as

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{array}$$

Assume that both (P) and (D) are feasible. Let x^* be an optimal solution to (P) and let y^* be an optimal solution to (D).

Associated with (P), consider a two-stage stochastic program (P_p) defined as

$$(P_p) \quad \begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i d_i^T u_i \\ \text{subject to} & Ax = b, \\ & p_i T_i x + p_i W_i u_i = p_i h_i, \quad i = 1, \dots, N, \\ & x \geq 0, \\ & u_i \geq 0, \quad i = 1, \dots, N, \end{array}$$

where $A \in I\!\!R^{m \times n}$, $b \in I\!\!R^m$, $c \in I\!\!R^n$, $p_i \in I\!\!R$, $T_i \in I\!\!R^{m_i \times n}$, $W_i \in I\!\!R^{m_i \times n_i}$, $h_i \in I\!\!R^{m_i}$, $d_i \in I\!\!R^{n_i}$, $i = 1, \dots, N$. The variables in (P_p) are thus $x \in I\!\!R^n$ and $u_i \in I\!\!R^{n_i}$, $i = 1, \dots, N$.

Assume that $p_i > 0$, $i = 1, \dots, N$, $\sum_{i=1}^N p_i = 1$, and in addition assume that $d_i \geq 0$, $i = 1, \dots, N$. Finally assume that (P_p) is feasible.

(a) Derive a dual linear program, (D_p), associated with (P_p). . . . . (3p)

(b) Give a feasible solution to (D_p). Make use of the known optimal solutions to (P) and (D). . . . . (2p)

(c) Show that $\text{optval}(P_p) \geq \text{optval}(P)$. Make the argument based on comparing (P_p) and (P). . . . . (2p)

(d) Once again, show that $\text{optval}(P_p) \geq \text{optval}(P)$. This time, make use of the feasible solution of (3b) in your argument. . . . . (3p)

4. Consider the integer program (IP) defined as

$$(IP) \quad \begin{array}{ll} \text{minimize} & -x_1 - 3x_3 - x_4 \\ \text{subject to} & -4x_1 - 5x_2 - 6x_3 - 7x_4 \geq -10, \\ & -x_1 - x_2 \geq -1, \\ & -x_3 - x_4 \geq -1, \\ & x_j \in \{0, 1\}, \quad j = 1, \dots, 4. \end{array}$$

Assume that the constraints $-x_1 - x_2 \geq -1$ and $-x_3 - x_4 \geq -1$ are relaxed with corresponding nonnegative multipliers v_1 and v_2. Let $\varphi(v)$ denote the resulting dual objective function. Finally, let $\widehat{v} = (1\ 2)^T$.

(a) Calculate $\varphi(\widehat{v})$. The corresponding Lagrangian relaxed problem for $v = \widehat{v}$ may be solved in any way, that need not be systematic. Give all optimal solutions to the Lagrangian relaxed problem for $v = \widehat{v}$. . . . . (6p)

(b) Use your result of (4a) to give two subgradients to φ at $\widehat{v}$. . . . . (3p)

(c) Use your result of (4b) to show that $\widehat{v}$ is an optimal solution to the dual problem. . . . . (1p)

5. Consider a cutting-stock problem with the following data:

$$W = 14, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 7, \quad b = \begin{pmatrix} 40 & 90 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

Solve the LP-relaxed problem associated with the above problem. Use the pure cut patterns to create an initial basic feasible solution, i.e., create one cut pattern with only w_1-rolls and correspondingly for w_2 and w_3.

You may utilize the fact that the subproblems that arise are small, and they may be solved in any way, that need not be systematic. We suggest that you do not use dynamic programming but instead solve the subproblem by enumeration and in case of non-unique solution selects the one with the most w_2-rolls. (As the requirement for w_2-rolls is the significantly largest.)

Finally create a "good" solution to the original problem based on your solution to the LP-relaxed problem. Comment on the quality of this solution. .... (10p)

Good luck!