KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday January 13 2011 8.00–13.00

Examiner: Anders Forsgren.

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Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (LP) defined as

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 12 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & -3 & 0 & 0 \end{pmatrix}^T.$$

Let $\widehat{x} = (5\ 2\ 3\ 0)^T$.

(a) Show that $\widehat{x}$ is not a basic feasible solution. .... (1p)

(b) Starting from $\widehat{x}$, find a basic feasible solution $\widetilde{x}$ such that $c^T\widetilde{x} \leq c^T\widehat{x}$. .... (3p)

(c) Solve (LP) by a suitable method, starting at $\widetilde{x}$. .... (6p)

2. Consider the stochastic program (P) given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \ldots, \omega_N$ with corresponding probabilities $p_1, \ldots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \dots, N, \\ & x \ge 0, \\ & y_i \ge 0, \quad i = 1, \dots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

3. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \ge 0, \end{array}$$

where

$$A = \begin{pmatrix} 6 & 2 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 12 & 3 & 3 & 6 & 3 \end{pmatrix}^T.$$

(a) AF has solved (PLP) by the simplex method. He has then obtained solutions $\widehat{x} = (1\ \ 1\ \ 0\ \ 0\ \ 0)^T$, $\widehat{y} = (2\ \ -1)^T$, and $\widehat{s} = (0\ \ 0\ \ 0\ \ 3\ \ 3)^T$. Verify that these solutions are optimal to (PLP) and (DLP) respectively. .... (2p)

(b) AF has then implemented a primal-dual interior method in Matlab. To test his solver, he has solved the primal-dual nonlinear equations accurately for $\mu = 10^{-4}$. He has then obtained the following approximate numbers for $x(\mu)$, $y(\mu)$, and $s(\mu)$:

```
xmu' =
    0.3924    2.2152    1.2153    0.0000    0.0000

ymu' =
    2.0000   -1.0000

smu' =
    0.0003    0.0000    0.0001    3.0000    3.0000
```

AF has solved the equations as accurately as possible, and he is confused. The values of $y(\mu)$ and $s(\mu)$ behave as he expects, they are near $\widehat{y}$ and $\widehat{s}$ respectively. However, the values of $x(\mu)$ are nowhere near $\widehat{x}$. Explain the situation to AF. Do this by using the information given in (3a) to characterize all optimal solutions to (PLP) and show that $x(\mu)$ is in fact close to an optimal solution. .... (8p)

4. Consider the integer program (IP) defined as

$$
(IP) \qquad
\begin{array}{ll}
\text{minimize} & -x_1 - 3x_3 - x_4 \\
\text{subject to} & -4x_1 - 5x_2 - 6x_3 - 7x_4 \geq -10, \\
& -x_1 - x_2 \geq -1, \\
& -x_3 - x_4 \geq -1, \\
& x_j \in \{0,1\}, \quad j = 1,\ldots,4.
\end{array}
$$

Assume that the constraint $-4x_1 - 5x_2 - 6x_3 - 7x_4 \geq -10$ is relaxed with corresponding nonnegative multiplier u. Let $\varphi(u)$ denote the resulting dual objective function.

(a) Express $\varphi(u)$ in terms of a suitable optimization problem. .... (4p)

(b) Show that $u \in [0, 1/4]$ is optimal to the resulting dual problem. You need not use a systematic method and you may utilize the fact that the dual problem is one-dimensional. .... (4p)

(c) What can be said about the bound on the optimal value of this particular (IP) given by the optimal value of the dual problem compared to the bound given by the linear programming relaxation? .... (2p)

5. Consider the optimization problem (P) given by

$$
(P) \qquad
\begin{array}{ll}
\text{minimize} & 3x_1 + 4x_2 + 5x_3 + 4x_4 \\
\text{subject to} & 4x_1 + x_2 + 3x_3 + 2x_4 = 2, \\
& |x_1| + |x_2| + |x_3| + |x_4| \leq 1.
\end{array}
$$

Problem (P) may be reformulated as a linear program. We will take an alternative approach. Your task is to solve (P) using Dantzig-Wolfe decomposition taking into account problem structure.

(a) Initially, consider the optimization problem

$$
(P_1) \qquad
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{n} v_j x_j \\
\text{subject to} & \sum_{j=1}^{n} |x_j| \leq 1,
\end{array}
$$

where v_j, $j = 1,\ldots,n$, are known coefficients.

Show that (P_1) may be reformulated as the linear program

$$
(LP_1) \qquad
\begin{array}{ll}
\text{minimize} & -\sum_{j=1}^{n} |v_j| y_j \\
\text{subject to} & \sum_{j=1}^{n} y_j \leq 1, \\
& y_j \geq 0, \quad j = 1,\ldots,n.
\end{array}
$$

Finally, let k be an index such that $|v_k| \geq |v_j|$, $j = 1,\ldots,n$. Show that $y_k = 1$, $y_j = 0$, $j = 1,\ldots,k-1,k+1,\ldots,n$, are optimal to (LP_1), and $x_k = -\operatorname{sign}(v_k)$, $x_j = 0$, $j = 1,\ldots,k-1,k+1,\ldots,n$, are optimal to (P_1). .... (3p)

(b) Now return to solving (P) by Dantzig-Wolfe decomposition. Consider the equality constraint $4x_1 + x_2 + 3x_3 + 2x_4 = 2$ as the hard constraint. For $S = \left\{x \in I\!R^4 : \sum_{j=1}^4 |x_j| \le 1\right\}$, write $x \in S$ as a convex combination of the extreme points of S. In the master problem, start with the basis that corresponds to the extreme points $(1\ \ 0\ \ 0\ \ 0)^T$ and $(0\ \ -1\ \ 0\ \ 0)^T$. The results of (5a) may be used to solve the subproblem(s) that arise(s). .... (7p)

Good luck!