KTH Mathematics

SF2812 Applied linear optimization, final exam
Thursday January 10 2013 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider a transportation problem (TP) defined as

$$(TP)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}x_{ij} \\ \text{subject to} & \displaystyle\sum_{j=1}^{4} x_{ij} = a_i, \quad i = 1,2,3, \\ & \displaystyle\sum_{i=1}^{3} x_{ij} = b_j, \quad j = 1,2,3,4, \\ & x_{ij} \geq 0, \quad i = 1,2,3, \ j = 1,2,3,4, \end{array}$$

where

$$C = \begin{pmatrix} 4 & 2 & 5 & 1 \\ 7 & 4 & 7 & 5 \\ 6 & 4 & 6 & 2 \end{pmatrix}, \quad a = \begin{pmatrix} 8 \\ 12 \\ 10 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 8 \\ 7 \\ 9 \end{pmatrix}.$$

The dual problem associated with (TP) may be written as

$$(DTP)\quad \begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{3} a_i u_i + \sum_{j=1}^{4} b_j v_j \\ \text{subject to} & u_i + v_j \leq c_{ij}, \quad i = 1,2,3, \quad j = 1,2,3,4. \end{array}$$

You have been given $\widehat{X}$, $\widehat{u}$ and $\widehat{v}$ as

$$\widehat{X} = \begin{pmatrix} 6 & 1.5 & 0 & 0.5 \\ 0 & 6.5 & 5.5 & 0 \\ 0 & 0 & 1.5 & 8.5 \end{pmatrix}, \quad \widehat{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \widehat{v} = \begin{pmatrix} 3 \\ 1 \\ 4 \\ 0 \end{pmatrix}.$$

(a) A friend of yours, who has not taken this course, claims that $\widehat{X}$ cannot be optimal to (TP), since the transportation problem should have integer valued optimal solutions when a and b are integers. Comment on your friend's claim. .... (2p)

(b) Verify that $\widehat{X}$ is optimal to (TP) and that $\widehat{u}$, $\widehat{v}$ is optimal to (DTP). ... (3p)
Hint: With S given by $s_{ij} = c_{ij} - u_i - v_j$, $i = 1,2,3$, $j = 1,2,3,4$, it holds that

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

(c) Find, using $\widehat{X}$, two integer valued optimal solutions to (TP). .... (3p)
Hint: It holds that $\sum_{i=1}^{3}\sum_{j=1}^{4} c_{ij}w_{ij} = 0$, $\sum_{j=1}^{4} w_{ij} = 0$, $i = 1,2,3$, and $\sum_{i=1}^{3} w_{ij} = 0$, $j = 1,2,3,4$, for

$$W = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(d) Explain why you would not obtain $\widehat{X}$ as an answer if you used the simplex method to solve (TP). .... (2p)

2. Consider fitting a line $y = kx + l$ to a number of given points (x_i, y_i), $i = 1, \dots, m$. In particular, k and l should be chosen so that the maximum deviation in the y-direction is minimized, i.e., k and l are chosen according to $\min_{k,l} \max_{i=1,\dots,m} |kx_i + l - y_i|$. By introducing the extra variable z, the problem may be written as an LP problem on the form

$$(LP) \quad \begin{array}{ll} \text{minimize} & z \\ \text{subject to} & -z \le kx_i + l - y_i \le z, \quad i = 1, \dots, m, \end{array}$$

where x_i, $i = 1, \dots, m$ and y_i, $i = 1, \dots, m$, are given parameters, and k, l and z are the variables. We assume that $m \ge 3$ and $x_i \ne x_j$ for $i \ne j$.

(a) Formulate the dual problem (DLP) associated with (LP). .... (5p)

(b) Given an optimal solution k, l, z to this line fitting problem, show that there are at least three points among the given points (x_i, y_i), $i = 1, \dots, m$, at which $|kx_i + l - y_i| = z$. You should motivate this result based on properties of the linear programs. .... (5p)

3. Let (P) and (D) be defined by

$$(P) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \ge 0, \end{array} \qquad \text{and} \qquad (D) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \ge 0. \end{array}$$

For a fixed positive barrier parameter μ, consider the primal-dual nonlinear equations

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

where we in addition require $x > 0$ and $s > 0$. Here, $X = \operatorname{diag}(x)$, $S = \operatorname{diag}(s)$ and e is an n-vector with all components one.

(a) Assume that $x(\mu)$, $y(\mu)$ and $s(\mu)$ solve the primal-dual nonlinear equations for a given positive μ, with $x(\mu) > 0$ and $s(\mu) > 0$. Show that $x(\mu)$ is feasible to (P) and $y(\mu), s(\mu)$ are feasible to (D) with duality gap $n\mu$. ....(3p)

(b) Derive the system of linear equations that results when the primal-dual nonlinear equations are solved by Newton's method. .... (5p)

(c) How are the implicit constraints $x > 0$ and $s > 0$ handled in a Newton-based interior method that approximately solves the primal-dual system of nonlinear equations for a sequence of decreasing values of μ? ....(2p)

4. Consider the binary integer programming problem (IP) given by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -5x_1 - 7x_2 - 10x_3 \\ \text{subject to} & -3x_1 - 6x_2 - 7x_3 \geq -8, \\ & -x_1 - 2x_2 - 3x_3 \geq -3, \\ & x_j \in \{0, 1\}, \quad j = 1, \dots, n. \end{array}$$

Assume that the constraint $-3x_1 - 6x_2 - 7x_3 \geq -8$ is relaxed by Lagrangian relaxation for a nonnegative multiplier u.

(a) For $u = 1$, compute two optimal solutions to the resulting Lagrangian relaxed problem. The Lagrangian relaxed problem may be solved by any method, that need not be systematic. ....(4p)

(b) Use the two optimal solutions to the Lagrangian relaxed problem computed in Exercise 4a to give two different subgradients to the dual objective function φ at $u = 1$. .... (4p)

(c) Show that there is a convex combination of the two subgradients computed in Exeercise 4b that gives the zero vector. What is the implication for the dual problem? ....(2p)

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \dots, m$. The demand for roll i is given by b_i, $i = 1, \dots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(3\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!