SF2812 Applied linear optimization, final exam
Thursday May 22 2014 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Let (P) and (D) be defined by

$$(P)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad \text{and} \qquad (D)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

(a) Let x be a feasible solution to (P) and let y, s be a feasible solution to (D). Show that the duality gap for these solutions is given by $x^T s$ and motivate the conclusion that we have optimal solutions for the two problems if and only if $x_j \cdot s_j = 0$ for all j. .... (4p)

(It may be assumed known that if (P) has an optimal solution, then (D) has an optimal solution, and the optimal values are equal.)

(b) Show that if (P) has an optimal solution, then there is at least one extreme point (basic feasible solution) which is optimal. .... (6p)

(You may for example use the representation theorem without proof.)

2. Consider the linear program (LP) defined as

$$(LP)\quad \begin{array}{ll} \min & x_1 + 3x_2 \\ \text{då} & x_1 + x_2 = 1, \\ & x_1 \geq 0, \ x_2 \geq 0. \end{array}$$

(a) For a fixed positive barrier parameter μ, formulate the primal-dual system of nonlinear equations corresponding to the problem above. Use the fact that the problem is small to give explicit expressions for the solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the system of nonlinear equations. .... (8p)

Hint: You may for example find an explicit expression for $y(\mu)$ and then express $x(\mu)$ and $s(\mu)$ in terms of this $y(\mu)$.

(b) Calculate optimal solutions to (LP) and the corresponding dual problem by letting $\mu \to 0$ in the expressions given in (2a). Verify optimality. .... (2p)

3. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 & 1 & -1 & c_4 \end{pmatrix}^T.$$

Are there values of b_1, b_2, c_1 and c_4 such that $\widehat{x} = (3\ 2\ 1\ 0)^T$ is optimal to (LP)? If so, determine all such values. .... (10p)

Hint: It holds that $Av = 0$ for $v = (1\ \ -2\ \ 1\ \ 0)^T$.

4. Consider the integer linear programming problem

$$\begin{array}{ll} \text{minimize} & -\sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} + \sum_{j=1}^n f_j z_j \\ \text{subject to} & \sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n, \\ & \sum_{i=1}^n a_i x_{ij} \leq b_j z_j, \quad j = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, n, \ j = 1, \dots, n, \\ & z_j \in \{0, 1\}, \quad j = 1, \dots, n, \end{array}$$

where a_i, $i = 1, \dots, n$, b_j, $j = 1, \dots, n$, f_j, $j = 1, \dots, n$, and c_{ij}, $i = 1, \dots, n$, $j = 1, \dots, n$, are nonnegative integer constants.

(a) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{j=1}^n x_{ij} = 1, \quad i = 1, \dots, n,$$

are relaxed by Lagrangian relaxation. .... (2p)

(b) Formulate the Lagrangian relaxed problem that arises when the constraints

$$\sum_{i=1}^n a_i x_{ij} \leq b_j z_j, \quad j = 1, \dots, n,$$

are relaxed by Lagrangian relaxation. .... (2p)

(c) Describe how the Lagrangian relaxed problems can be solved in the two cases. .... (4p)

(d) Discuss which of the two Lagrangian relaxations that would give the best underestimate of the optimal value of the original problem when the corresponding dual problem is solved. .... (2p)

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(3\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!