SF2812 Applied linear optimization, final exam
Wednesday June 7 2017 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (LP) and its dual (DLP) defined as

$$(LP) \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP) \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 5 & -1 & 3 & 2 & 4 \\ 0 & 2 & 0 & 4 & 0 & 3 \\ -1 & 2 & 3 & 0 & 2 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \\ 1 \end{pmatrix},$$
$$c = \begin{pmatrix} 1 & 11 & 1 & 2 & 8 & 10 \end{pmatrix}^T.$$

The related barrier transformed problem (P_μ), defined by

$$(P_\mu) \quad \begin{array}{ll} \text{minimize} & c^T x - \mu \sum_{j=1}^{6} \ln x_j \\ \text{subject to} & Ax = b, \\ & (x > 0), \end{array}$$

has an optimal solution $\widetilde{x}$ and lagrange multiplier vector $\widetilde{y}$ for $\mu = 10^{-3}$ which numerically is given by approximately

```
xtilde =
    1.9906
    0.0010
    0.9956
    0.9993
    0.0005
    0.0003

ytilde =
    1.9987
   -0.9993
    0.9992
```

(a) Use the above numbers to give an approximate solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the primal-dual nonlinear equations, associated with a primal-dual interior method for solving (LP), for $\mu = 10^{-3}$. . . . . (4p)

Hint: The vector of componentwise inverses of $\widetilde{x}$ is numerically approximately given by

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1./xtilde = 1.0e+03 *
    0.0005
    1.0064
    0.0010
    0.0010
    2.0040
    3.0045
```

(b) The above problem (LP) has an optimal solution which is integer valued, and there is an optimal solution to (DLP) for which y and s are integer valued. Given this knowledge, use your results from Question 1a to make a qualified guess of optimal solutions to (LP) and (DLP) respectively. Motivate your guess and verify optimality. . . . . (4p)

(c) If the simplex method had been used to solve (LP), would the same primal optimal solution have been obtained? Comment on the result. . . . . (2p)

Hint: It holds that

$$\det \begin{pmatrix} 1 & -1 & 3 \\ 0 & 0 & 4 \\ -1 & 3 & 0 \end{pmatrix} \neq 0.$$

2. Consider the linear program

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}^T.$$

(a) Show that the basis given by $\mathcal{B} = \{2, 3\}$ gives a corresponding dual basic solution which is feasible to the dual problem associated with (LP). . . . . .(3p)

(b) Solve (LP) by the dual simplex method, starting from the basis given in Question 2a. . . . .(7p)

3. Consider the stochastic program (P) given by

$$(P) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & T(\omega)x = h(\omega), \\ & x \geq 0, \end{array}$$

where ω is a stochastic variable and $T(\omega)x = h(\omega)$ is to be interpreted as an "informal" stochastic constraint. Assume that ω takes on a finite number of values $\omega_1, \dots, \omega_N$ with corresponding probabilities $p_1, \dots, p_N$. Let T_i denote $T(\omega_i)$ and let h_i denote $h(\omega_i)$.

(a) Explain how the deterministically equivalent problem

$$\begin{array}{ll} \text{minimize} & c^T x + \sum_{i=1}^{N} p_i q_i^T y_i \\ \text{subject to} & Ax = b, \\ & T_i x + W y_i = h_i, \quad i = 1, \dots, N, \\ & x \geq 0, \\ & y_i \geq 0, \quad i = 1, \dots, N, \end{array}$$

arises. (We assume, for simplicity, "fix compensation", i.e., W does not depend on i.) .... (6p)

(b) Define *VSS* in terms of suitable optimization problems. .... (2p)

(c) Define *EVPI* in terms of suitable optimization problems. .... (2p)

4. Consider the integer programming problem (IP) given by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & -3x_1 - 4x_2 - 3x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \geq 1, \\ & -x_1 - 2x_2 - 3x_3 \geq -2, \\ & x_j \geq 0, \ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{array}$$

For $u \in I\!R$, let

$$\varphi(u) = \begin{array}{ll} \text{minimize} & -3x_1 - 4x_2 - 3x_3 - u(x_1 + x_2 + x_3 - 1) \\ \text{subject to} & -x_1 - 2x_2 - 3x_3 \geq -2, \\ & x_j \geq 0, \ x_j \text{ integer}, \quad j = 1, \dots, 3. \end{array}$$

You may throughout this question use the fact that the problem is small and your methods for solving subproblems that arise need not by systematic.

(a) Determine $\varphi(u)$ for $u \in I\!R$. .... (3p)

(b) Your friend AF is a bit confused. By inspection, he can see that an optimal solution to (IP) is given by $x = (2 \ 0 \ 0)^T$ so that $\text{optval}(IP) = -6$, where $\text{optval}(IP)$ denotes the optimal value of (IP). By his calculations, he has $\varphi(-1) = -5$. Explain to him why it is not a contradiction to our theory on Lagrangian relaxation that there exists a $u \in I\!R$ such that $\varphi(u) > \text{optval}(IP)$. .... (3p)

(c) Determine an optimal solution to the dual problem that results when the constraint $x_1 + x_2 + x_3 \geq 1$ is relaxed by Lagrangian relaxation. In addition, determine the duality gap. .... (4p)

5. Consider a cutting-stock problem with the following data:

$$W = 11, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook. Given are rolls of width W. Rolls of m different widths are demanded. Roll i has width w_i, $i = 1, \ldots, m$. The demand for roll i is given by b_i, $i = 1, \ldots, m$. The aim is to cut the W-rolls so that a minimum number of W-rolls are used.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(3\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!