SF2812 Applied linear optimization, final exam
Tuesday June 5 2018 14.00–19.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one exercise per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

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1. Consider the linear programming problem (PLP) and its dual (DLP) defined as

$$(PLP)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

In the discussion below, we let $\text{optval}(PLP) = \infty$ if (PLP) is infeasible and analogously $\text{optval}(DLP) = -\infty$ if (DLP) is infeasible, where "optval" denotes the optimal value.

Assume that $\widetilde{y}$, $\widetilde{s}$ is a feasible solution to (DLP).

(a) Give a lower bound on the optimal value of (DLP). ....(2p)

(b) Give a lower bound on the optimal value of (PLP). Is (PLP) necessarily feasible? ....(2p)

(c) Assume that there exists $\eta \in I\!R^m$ and $q \in I\!R^n$ such that $A^T\eta + q = 0$, $q \geq 0$ and $b^T\eta > 0$. What is the implication on (PLP) and (DLP)? ....(3p)

(d) Assume that $\widetilde{x}$ is an optimal solution to (PLP) and in addition assume that $\widetilde{x}^T\widetilde{s} = 1$. Is it possible that $\widetilde{y}$, $\widetilde{s}$ is an optimal solution to (DLP)? ....(3p)

2. Let (LP) and its dual (DLP) be defined as

$$(LP)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \quad \text{and} \quad (DLP)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ -1 & 4 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 2 \\ 7 \\ 8 \end{pmatrix}, \quad \text{and}$$

$$c = \begin{pmatrix} 3 & -1 & 1 & 2 & 1 & 4 \end{pmatrix}^T.$$

(a) A person named AF has used GAMS to model and solve this problem. AF has been told that he can solve either (LP) or (DLP) for finding the optimal solutions to (LP) and (DLP). He has chosen to solve (LP). The GAMS input file can be found at the end of the exam, and a partial GAMS output file reads:

```
               S O L V E      S U M M A R Y

     MODEL   lpex                OBJECTIVE  objval
     TYPE    LP                  DIRECTION  MINIMIZE
     SOLVER  CPLEX               FROM LINE  41

**** SOLVER STATUS     1 Normal Completion
**** MODEL STATUS      1 Optimal
**** OBJECTIVE VALUE                4.0000

                   LOWER     LEVEL     UPPER    MARGINAL

---- EQU obj         .         .         .       -1.000

  obj  objective function

---- EQU cons  constraints

      LOWER     LEVEL     UPPER    MARGINAL

i1     7.000     7.000     7.000     1.000
i2     2.000     2.000     2.000    -1.000
i3     7.000     7.000     7.000     1.000
i4     8.000     8.000     8.000    -1.000

                   LOWER     LEVEL     UPPER    MARGINAL

---- VAR objval     -INF      4.000     +INF       .

  objval  objective function value

---- VAR x  primal variables

      LOWER     LEVEL     UPPER    MARGINAL

j1      .         .        +INF      2.000
j2      .        2.000     +INF       .
j3      .        1.000     +INF       .
j4      .        1.000     +INF       .
j5      .        3.000     +INF       .
j6      .         .        +INF      5.000
```

The only catch is that AF does not know how to extract the optimal solutions from the GAMS output. Help AF obtain the optimal solutions to (LP) and (DLP) from the GAMS output file. ....(4p)

(b) AF claims that if b_2 is changed to $2 + \delta$ and b_3 simultaneously is changed to $7 + \delta$, the optimal value is unchanged. Show that AF is right. Do so without solving any system of linear equations. ....(2p)

(c) Give bounds on δ for which the answer of Question 2b is valid. The system of linear equations that arises need not be solved in a systematic way. ..... (4p)

3. Consider the linear programming problem (LP) and its dual (DLP) defined as

$$(LP)\quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \qquad (DLP)\quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 3 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 5 \end{pmatrix},$$
$$c = \begin{pmatrix} 3 & 1 & 0 & 0 \end{pmatrix}^T.$$

The related barrier transformed problem (P_μ), defined by

$$(P_\mu)\quad \begin{array}{ll} \text{minimize} & c^T x - \mu \sum_{j=1}^{4} \ln x_j \\ \text{subject to} & Ax = b, \\ & (x > 0), \end{array}$$

has a feasible solution $\widetilde{x}$ which numerically is given by approximately

```
xtilde =
    0.0916
    4.9221
    0.1053
    9.8579
```

You have been given a matrix Z such that $AZ = 0$, as

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

In addition, the vector of componentwise inverses of $\widetilde{x}$ is numerically approximately given by

```
1./xtilde =
   10.9135
    0.2032
    9.4925
    0.1014
```

and the numerical values of this vector premultiplied by Z^T are numerically approximately given by

```
Z'*(1./xtilde) =
    30
    10
```

(a) Use the above information on $\widetilde{x}$ and Z to show that $\widetilde{x}$ is an approximate optimal solution to (P_μ) for $\mu = 0.1$ (up to numerical precision). . . . . (4p)
Hint: It may be helpful to use the fact that the nullspace of A and the range space of A^T are orthogonal complements that together span $I\!R^4$.

(b) The above problem (LP) has an optimal solution which is integer valued. Given this knowledge, use your results from Question 3a to make a qualified guess of optimal solution to (LP). Motivate your guess and verify optimality. Systems of linear equations that arise need not be solved in a systematic way. . . . . (4p)

(c) Starting from the result of Question 3a, give an approximate solution $x(\mu)$, $y(\mu)$ and $s(\mu)$ to the primal-dual nonlinear equations, associated with a primal-dual interior method for solving (LP), for $\mu = 0.1$. . . . . (2p)
Hint: The structure of A is such that no complicated system of linear equations need be solved.

4. Consider the integer program (IP) defined by

$$(IP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0, \quad x \text{ integer}. \end{array}$$

Let z_{IP} denote the optimal value of (IP).

Associated with (IP) we may define the dual problem (D) as

$$(D) \qquad \begin{array}{ll} \text{maximize} & \varphi(u) \\ \text{subject to} & u \geq 0, \end{array}$$

where $\varphi(u) = \min\{c^T x + u^T(b - Ax) : Cx \geq d, x \geq 0 \text{ integer}\}$. Let z_D denote the optimal value of (D).

Let (LP) denote the linear program obtained from (IP) by relaxing the integer requirement, i.e.,

$$(LP) \qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b, \\ & Cx \geq d, \\ & x \geq 0. \end{array}$$

Let z_{LP} denote the optimal value of (LP).

Show that $z_{IP} \geq z_D \geq z_{LP}$. . . . . (10p)

5. Consider a cutting-stock problem with the following data:

$$W = 12, \quad m = 3, \quad w_1 = 3, \quad w_2 = 5, \quad w_3 = 9, \quad b = \begin{pmatrix} 60 & 50 & 40 \end{pmatrix}^T.$$

Notation and problem statement are in accordance to the textbook.

Given are rolls of width W, referred to as *W-rolls* below, containing the raw material. Smaller rolls of m different widths are to be cut out of the W-rolls, where each such smaller roll i has width w_i, $i = 1, \ldots, m$, and the demand for each such smaller roll i is given by b_i, $i = 1, \ldots, m$.

The aim is to cut the W-rolls so that a minimum number of W-rolls are used. This is done by forming *cut patterns*, where a cut pattern is a specification of how many of each smaller roll i that are included in this particular cutting of a W-roll. A cut pattern is represented by a nonnegative integer vector $(a_1, a_2, \ldots, a_m)$, where a_i specifies how many of the smaller roll i, $i = 1, \ldots, m$, that are included in the particular cut pattern.

(a) Solve the the LP-relaxed problem associated with the above problem. Start with the basic feasible solution associated with the three "pure" cut patterns $(4\ 0\ 0)^T$, $(0\ 2\ 0)^T$ and $(0\ 0\ 1)^T$. The subproblems that arise may be solved in any way, that need not be systematic. . . . . (8p)

(b) Determine a "near-optimal" solution to the original problem. Give a bound on the maximum deviation from the optimal value of the original problem. . . (2p)

Good luck!

GAMS file for Question 2:

```
sets
i rows          / i1*i4 /
j columns       / j1*j6 /;

table A(i,j)    values of the blocks
      j1    j2    j3    j4    j5    j6
i1     1     1     1     1     1
i2     2     1     1    -1
i3     1     3     1
i4    -1     4                       1 ;

parameter b(i)
          / i1   7
            i2   2
            i3   7
            i4   8 /;

parameter c(j)
          / j1   3
            j2  -1
            j3   1
            j4   2
            j5   1
            j6   4 /;

variables
        objval  objective function value
        x(j)    primal variables;

positive variable x;

equations
        obj             objective function
        cons(i)         constraints;

obj .. sum(j,c(j)*x(j)) =e= objval;
cons(i) .. sum(j,A(i,j)*x(j)) =e= b(i);

model lpex / all /;

solve lpex using lp minimizing objval;
```