SF2822 Applied nonlinear optimization, final exam
Tuesday June 3 2013 8.00–13.00

Examiner: Anders Forsgren, tel. 08-790 71 27.

Allowed tools: Pen/pencil, ruler and eraser. *Note!* Calculator is not allowed.

Solution methods: Unless otherwise stated in the text, the problems should be solved by systematic methods, which do not become unrealistic for large problems. Motivate your conclusions carefully. If you use methods other than what have been taught in the course, you must explain carefully.

Note! Personal number must be written on the title page. Write only one question per sheet. Number the pages and write your name on each page.

22 points are sufficient for a passing grade. For 20-21 points, a completion to a passing grade may be made within three weeks from the date when the results of the exam are announced.

---

1. Consider the nonlinear program

$$(NLP) \quad \begin{array}{ll} \displaystyle\operatorname*{minimize}_{x\in I\!R^3} & f(x) \\ \text{subject to} & a^T x - b = 0, \\ & g(x) \geq 0, \end{array}$$

where $a = (0\ 1\ 0)^T$, $b = 1$, and it holds that $f : I\!R^3 \to I\!R$ and $g : I\!R^3 \to I\!R$ are twice-continuously differentiable. For $x^* = (1\ 1\ 1)^T$, it is known that

$$f(x^*) = 0, \quad \nabla f(x^*) = \begin{pmatrix} -1 & -2 & 0 \end{pmatrix}^T, \quad \nabla^2 f(x^*) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$g(x^*) = 0, \quad \nabla g(x^*) = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T.$$

(a) Does x^* satisfy the first-order necessary optimality conditions for (NLP)? (3p)

(b) Are there conditions on $\nabla^2 g(x^*)$, which guarantee that x^* is a local minimizer to (NLP)? If so, which conditions? ....(4p)

(c) Are there conditions on $\nabla^2 g(x)$, $x \in I\!R^3$, which guarantee that x^* is a global minimizer to (NLP)? If so, which conditions? These conditions should not include any properties related to f, a or b. ....(3p)

2. Consider the strictly convex bound-constrained quadratic program (QP) given by

$$(QP) \quad \begin{array}{ll} \displaystyle\operatorname*{minimize}_{x\in I\!R^3} & \frac{1}{2}x^T H x + c^T x \\ \text{subject to} & x \geq 0, \end{array}$$

where

$$H = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix}, \quad c = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

Solve (QP) using an active-set method. Start at the point $x = (0\ 0\ 1)^T$ and let the constraints $x_1 \geq 0$ and $x_2 \geq 0$ be active in the first iteration. The linear equations that arise may be solved in any way that need not be systematic. Motivate each step carefully. .... (10p)

Hint: You may find the following relationship helpful:

$$\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 1 \end{pmatrix}.$$

3. Consider the same quadratic program (QP) as in Question 2.

Assume that we want to solve (QP) with a primal-dual interior point method. Also assume that we initially choose $x^{(0)} = (2\ 1\ 2)^T$, $\lambda^{(0)} = (1\ 2\ 1)^T$, and $\mu = 0.2$.

(a) When the constraints are in the form $Ax \geq b$, one may introduce slack variables s and rewrite the constraints as $Ax - s = b$, $s \geq 0$, when applying the interior method. Explain why this is not necessary for the given initial value $x^{(0)}$. (2p)

(b) Formulate the linear system of equations to be solved in the first iteration of the primal-dual interior point method for the given initial values. Formulate the general form and then introduce explicit numerical values into the system of equations. .... (5p)

(c) If the linear system of equations of Question 3b are solved, and the steps in the x-direction and the λ-direction are denoted by Δx and $\Delta\lambda$ respectively, one obtains

$$\Delta x \approx \begin{pmatrix} 0.3455 \\ -1.0455 \\ -0.6182 \end{pmatrix}, \quad \Delta\lambda \approx \begin{pmatrix} -1.0727 \\ 0.2909 \\ -0.5909 \end{pmatrix}.$$

Explain why it is not suitable to use the unit step, i.e, why it is not suitable to let $x^{(1)} = x^{(0)} + \Delta x$ and $\lambda^{(1)} = \lambda^{(0)} + \Delta\lambda$. Also explain how you would choose $x^{(1)}$ and $\lambda^{(1)}$. You need not give precise numerical values of $x^{(1)}$ and $\lambda^{(1)}$, but you should explain the principle. .... (3p)

4. Derive the expression for the symmetric rank-1 update, C_k, in a quasi-Newton update $B_{k+1} = B_k + C_k$. .... (10p)

5. Consider the optimization problem

$$(P) \qquad \underset{x \in I\!R^n}{\text{minimize}} \; \{ \max_{i=1,\dots,m} f_i(x) \},$$

where the functions f_i, $i = 1, \dots, m$, are twice continuously differentiable and convex on $I\!R^n$. A drawback of the formulation given by (P) is that the objective function $\max_{i=1,\dots,m} f_i(x)$ is nondifferentiable.

(a) Rewrite (P) as an equivalent convex nonlinear program, where the objective function and constraint functions are differentiable. Motivate convexity. . . (3p)

Hint: It may be helpful to think of how the problem would be formulated in GAMS, where there is no explicit objective function.

(b) Assume that you want to solve the nonlinear program that you have formulated by a sequential quadratic programming method. Formulate the quadratic programming subproblem in explicit form. . . . . (5p)

(c) The nonlinear programming problem is a convex problem. Can you ensure that the quadratic programming subproblem is a convex problem? . . . . (2p)

Good luck!