KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part one: 14.00-16:00, January 8, 2021

Examiner: Xiaoming Hu, tel. 0707967831.

Important! This exam consists of two parts. The second part starts at 16:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts!

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory (pdf or paper version), Per Enqvist, Exercises in Mathematical Systems Theory (pdf or paper version), your own class notes (digital or paper version), and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.

The sub-problems in each problem are listed by the ascending order of difficulty whenever it is possible.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point unless otherwise indicated.**

(a) Consider an n-dimensional *time-varying* system $\dot{x}(t) = A(t)x(t)$, where $A(t)$ is continuous. If $\|x(t)\|^2 = \|x(t_0)\|^2$ $\forall t > t_0 \in R$ and $\forall x(t_0) \in R^n$, then $\Phi(s,t) = \Phi^T(t,s)$, where $\Phi(t,s)$ is the state transition matrix. .... (6p)

Answer: True, since $\|x(t)\|^2 = x^T(t_0)\Phi^T(t,t_0)\Phi(t,t_0)x(t_0)$, which implies that $\Phi^T(t,t_0)\Phi(t,t_0) = I$.

(b) Consider $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R^m,\ y \in R^p$. Which of the following statements are true? Multiple choices are possible and no reasoning is needed. .... (6p)

b1. If $rank\ A < n-1$, then (A,B) can never be controllable.

b2. We can design an observer to estimate $x(t)$ if (A,B,C) is a minimal realization.

b3. We can design observers such that $\|x(t) - \hat{x}(t)\| \le Ke^{-rt}$ for any given $r > 0$ and also solve the pole placement problem if and only if (A,B,C) is a minimal realization.

Answer: b2, b3.

(c) Consider again $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R^m,\ y \in R^p$ and $C \neq 0$. Assume that $dim\ ker\ \Omega \geq 1$. If $\forall x(0) \in ker\ \Omega$, the corresponding $x(t) \in ker\ \Omega\ \forall t \geq 0$ no matter what $u(t)$ we choose, then (A, B) is not reachable. (4p)

Answer: True, since we can not move a state in $ker\ \Omega$ to outside of $ker\ \Omega$ and $ker\ \Omega$ is strictly contained in R^n.

(d) Consider the optimal control problem for $\dot{x} = Ax + Bu,\ x(0) = x_0$:

$$\min_u \int_0^{t_1} (x^T C^T C x + u^T R u)dt + x^T(t_1) S x(t_1),$$

where $S \geq 0$ and $R > 0$. If $C = 0$, then $P(t)$ can never be positive definite for any $0 \leq t < t_1$, where $P(t)$ is the solution to the corresponding dynamical Riccati equation. ....(4p)

Answer: False. By solving P(t) using the adjoint system, we see that $P(t) > 0$ if $S > 0$.

2. (25p) Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \tag{1}$$

where $x \in R^3,\ u \in R,\ y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1}b = \frac{s^2 + s - 2}{(s^2 + 2s + 1)(s + \alpha)},$$

where α is a constant. Assume $c = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix}$.

(a) Find matrices A and b. .... (5p)

Answer: $A = [0\ 1\ 0; 0\ 0\ 1; -\alpha\ \ -(1+2\alpha)\ \ -(2+\alpha)]$, $b = [0; 0; 1]$.

(b) Is your (A, b) reachable? .... (3p)

Answer: Yes, ...

(c) What is the condition on α such that $\lim_{t\to\infty} e^{At}x_0 = 0$ for all $x_0 \in R^3$. . (4p)

Answer: $\alpha > 0$.

(d) For what α is (c, A) NOT observable? ....(5p)

Answer: $\alpha = 2, -1$

(e) Find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^3 : cx = 0\}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (3p)

Answer: $cx(t) \equiv 0$ implies $c\dot{x}(t) \equiv 0$, which is $-2x_2 + x_3 - \alpha x_1 - (1+2\alpha)x_2 - (2+\alpha)x_3 + u = 0$, and we can solve for u.

(f) Are the state trajectories $x(t)$ obtained in (e) bounded (you must prove your conclusion)? A solution $x(t)$ obtained in (e) means a solution to (1) when $u(t) = Kx$ is used and $cx(0) = 0$. .... (3p)

Answer: No, $x(t)$ is unbounded in general. by keeping $cx(t) \equiv 0$, the dynamics for the trajectories becomes $\dot{x}_1 = x_2,\ \dot{x}_2 = 2x_1 - x_2$, which is clearly unstable.

(g) For which α you find in (d) can we find a two dimensional minimal realization of the $r(s)$ such that for the minimal realization there exists $u(t) = Kx$ that makes both $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$ and $\lim_{t\to\infty} x(t) = 0$. .... (2p)

Answer: $\alpha = -1$, then the unstable zero is canceled.