KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
14.00-19:00, January 11, 2022

Examiner: Xiaoming Hu, tel. 0707967831.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.

The sub-problems in each problem are listed by the ascending order of difficulty whenever it is possible.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point.**

(a) Consider $\dot{x} = Ax$ where $x \in R^n$ and assume $A^T = -A$, then $x = 0$ is never asymptotically stable. .... (6p)

Answer: True, since $\|x(t)\| = \|x(0)\|$.

(b) Consider $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R^m,\ y \in R^p$.

b1. Assume that $rank\ A < n - 1$. Then $m \geq 2$ if (A, B) is controllable. ..(3p)

Answer: True, since for $m = 1$ a necessary condition for controllability is that $rank\ A \geq n - 1$.

b2. Let $m = p = 1$, we have $C(sI - A)^{-1}B = \frac{n(s)}{d(s)}$, where $d(s) = det\ (sI - A)$. If $n(s)$ and $d(s)$ do not share any common root then (A, B, C) is minimal. (3p)

Answer: True, since $\delta(R(s)) = n$ in this case.

(c) Assume (A, B) is controllable and (C, A) is observable. Then $(C, A + BK)$ is also observable for any K. .... (4p)

Answer: False, for example in the siso case we can find K such that there is zero/pole cancellation.

(d) Consider the optimal control problem for $\dot{x} = Ax + Bu,\ x(0) = x_0$:

$$\min_u \int_0^\infty (x^T C^T C x + u^T R u) dt,$$

where $R > 0$. If (A, B) is not controllable, then the optimal control does not exist. .... (4p)

Answer: False. A counter example is that $B = 0$, and A is a stable matrix. Then $u = 0$ is obviously the optimal control.

2. (25p) Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \tag{1}$$

where $x \in R^3,\ u \in R,\ y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1}b = \frac{s^2 + s - 2}{(s^2 + 2s + \alpha)(s + 1)},$$

where α is a constant. Assume $b = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

(a) Find matrices A and c. .... (5p)
Answer: A is in the standard controllable form and the last row is $(-\alpha\ \ -(2+\alpha)\ \ -3)$, $c = (-2\ 1\ 1)$.

(b) Is your (c, A) observable? .... (6p)
Answer: Observable if $\alpha \neq -3,\ 0$.

(c) Give the condition on α such that $\|e^{At}x_0\| \leq M\|x_0\|$ for all $x_0 \in R^3$ and all $t \geq 0$, where M is a positive constant. .... (4p)
Answer: $\alpha \geq 0$.

(d) Find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^3 : cx = 0\}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (3p)
Answer: $y = -2x_1 + x_2 + x_3 \equiv 0$ implies that $\dot{y} \equiv 0$ which gives $-2x_2 + x_3 - \alpha x_1 - (2+\alpha)x_2 - 3x_3 + u = 0$. From this we obtain the control.

(e) Are the state trajectories $x(t)$ obtained in (d) bounded (you must prove your conclusion)? A solution $x(t)$ obtained in (d) means a solution to (1) when $u(t) = Kx$ is used and $cx(0) = 0$. .... (4p)
Answer: No, since the trajectories contain e^t.

(f) For what α can we find a two dimensional minimal realization of the $r(s)$ such that for the minimal realization there exists $u(t) = Kx$ that makes both $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$ and $\lim_{t\to\infty} x(t) = 0$. .... (3p)
Answer: $\alpha = -3$, since in this case the unstable zero $s = 1$ will be cancelled.

3. (20p) Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+1} & \frac{1}{s+1} & \frac{\gamma+1}{s+1} \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} \end{bmatrix},$$

where γ is a real constant.

(a) Find the standard reachable realization. .... (6p)
Answer: Omitted.

(b) Compute the McMillan degree of $R(s)$. .... (6p)
Answer: $\delta(R) = 3$ if $\gamma \neq 1$; otherwise $\delta(R) = 2$.

(c) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer by computing $C(sI - A)^{-1}B$. . . . . (8p)

Answer: Since in $R(s)$ the first column= the second column, and the third is twice the first, the problem can be reduced to finding the minimal realization for

$$\begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix},$$

for which the standard controllable realization would do.

4. (20p) Consider the optimal control problem

$$\min_u J = \int_0^\infty (y^2 + \frac{1}{\epsilon}u^2)dt$$
$$\text{s.t.}$$
$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$x(0) = x_0,$$

where, $\epsilon > 0$, and

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(a) Show that the associated algebraic Riccati equation (ARE) has no positive definite solution if $a_1 \leq 0$. . . . . (4p)

Answer: Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, then we have $a_1p_1 - \epsilon p_2^2 = 0,\ (a_1 + a_2)p_2 + p_1 - \epsilon p_2 p_3 = 0$. Clearly $p_1 = 0$ if $a_1 \leq 0$.

(b) Let $P(\epsilon)$ denote the symmetric solution to the ARE. Show that $\lim_{\epsilon\to 0} \epsilon P(\epsilon)$ is positive definite if and only if $a_1 > 0$ and $a_2 > 0$. . . . . (6p)

Answer: Let $\bar{P} = \epsilon P$, then $A^T\bar{P} + \bar{P}A - \bar{P}BB^T\bar{P} + \epsilon Q = 0$. As $\epsilon \to 0$, we have $A^T\bar{P} + \bar{P}A - \bar{P}BB^T\bar{P} = 0$. Assume first that $\bar{P} > 0$, then we would have $\bar{P}^{-1}(-A^T) + (-A)\bar{P}^{-1} = -BB^T$. Since (A, B) is controllable, this Lyapunov equation has a positive definite solution iff $-A$ is a stable matrix. The rest follows.

(c) Show when $\lim_{\epsilon\to 0} \epsilon P(\epsilon) > 0$, $\lim_{\epsilon\to 0}(A - \epsilon BB^T P(\epsilon))$ has eigenvalues $\{-a_1, -a_2\}$. (5p)

Answer: From the Riccati equation we have $\bar{P}(A - BB^T\bar{P}) + A^T\bar{P} = 0$, thus $A - BB^T\bar{P} = -\bar{P}^{-1}A^T\bar{P}$.

(d) When $\lim_{\epsilon\to 0} \epsilon P(\epsilon) > 0$, show for any $\sigma > 0$, there exists $\epsilon_0 > 0$, such that for all $\epsilon < \epsilon_0$, $A - \sigma BB^T P(\epsilon)$ is a stable matrix. . . . . (5p)

Answer: By manipulating the ARE, we have $(A - \sigma BB^TP)^TP + P(A - \sigma BB^TP) = -Q - (2\sigma - \epsilon)PBB^TP$. Since $\lim_{\epsilon\to 0} \epsilon P(\epsilon) > 0$, $\exists\ \epsilon_0 < \sigma$, such that $P(\epsilon) > 0,\ \forall\epsilon \leq \epsilon_0$. Then by using similar technique used in (b) we draw the conclusion.

5. (15p)

(a) Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Show that for any $t_1 > 0$, $u = -B^T W^{-1}(t_1)x$ asymptotically stabilizes the system (namely $A - BB^T W^{-1}(t_1)$ is a stable matrix), where

$$W(t_1) = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt.$$

.... (8p)

Answer: We have $(A - BB^T W^{-1})W + W(A - BB^T W^{-1})^T = AW + WA^T - 2BB^T$. Let $L(t) = e^{-At}BB^T e^{-A^T t}$, then $\dot{L} = -Ae^{-At}BB^T e^{-A^T t} - e^{-At}BB^T e^{-A^T t}A^T$. By integrating both sides, we have $e^{-At_1}BB^T e^{-A^T t_1} - BB^T = -(AW + WA^T)$. Then $(A - BB^T W^{-1})W + W(A - BB^T W^{-1})^T = -BB^T - e^{-At_1}BB^T e^{-A^T t_1}$ and the conclusion follows.

(b) Consider the discrete-time matrix Riccati equation associated with Kalman filters

$$\begin{aligned} P(t+1) &= AP(t)A^T - AP(t)C^T[CP(t)C^T + DRD^T]^{-1}CP(t)A^T + BQB^T \\ P(0) &= P_0. \end{aligned}$$

Assume that $A = a,\ C = 1,\ D = 1,\ R = r > 0,\ Q = 0$ and $P_0 = p_0 > 0$, where $|a| \neq 1$.

(1) Show that $\lim_{t\to\infty} P(t)$ exists. .... (4p)
Answer: Since $P^{-1}(t+1) = \frac{1}{a^2}P^{-1}(t) + \frac{1}{a^2 r}$, we have $P^{-1}(t)$ converge to an equilibrium if $a^2 > 1$ and diverge to infinity if $a^2 < 1$. In the second case we have $P(t) \to 0$ and in the first case $P^{-1} = \frac{1}{a^2}P^{-1} + \frac{1}{a^2 r}$, which gives $\lim_{t\to\infty} P(t) = (a^2 - 1)r$.

(2) Denote $P = \lim_{t\to\infty} P(t)$, show that $|a - ak| < 1$, where $k = P(P + r)^{-1}$ is the Kalman gain. .... (3p)
Answer: if $|a| < 1$, we have $k = 0$ then $a - ak = a$; if $|a| > 1$, we have $k = \frac{a^2-1}{a^2}$, then $a - ak = \frac{1}{a}$.