CHAPTER 5

# Noninteracting control and tracking

In the previous chapter, we discussed the concepts of transmission zeros, relative degree, and zero dynamics from a geometric point of view. In this chapter we will study some more control problems in which these concepts play an important role in finding the solutions.

## 5.1. Noninteracting control

In this section we study the problem of noninteracting control. Early study of this problem can be found in [**13**], [**14**], [**16**].

Consider a square system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{5.1}$$

where

$$B = \begin{bmatrix} b_1 & \cdots & b_m \end{bmatrix}, \; C = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}.$$

PROBLEM 5.1 (Static noninteracting control). *Find $u = Fx + Gv$, such that the closed-loop system*

$$\begin{aligned} \dot{x} &= (A + BF)x + BGv \\ y &= Cx, \end{aligned} \tag{5.2}$$

*has some relative degree $(r_1, \ldots, r_m)$, and each output $y_i = c_i x$, $i = 1, \ldots, m$, is only influenced by the input $v_i$ and no other input.*

REMARK 5.1. *Note that $G$ must be invertible. The property that the output $y_i$ is not affected by the input $v_j$, if $j \neq i$, can be characterized as*

$$c_i(A + BF)^k (BG)_j = 0, \; \forall k \geq 0 \text{ where } i \neq j.$$

*The property that the closed-loop system has some relative degree eliminates trivial solutions, namely solutions in which some output is not affected by any input at all.*

This is the *restricted* problem of noninteracting control; restricted in the sense that only *static* state feedback is allowed.

THEOREM 5.1. *The static noninteracting control problem has a solution if and only if the MIMO system* (5.1) *has some relative degree $(r_1, \ldots, r_m)$.*

PROOF
Sufficiency: here we give a constructive proof. Since the system has relative degree $(r_1, \ldots, r_m)$, as discussed in the previous chapter, we can convert it into the normal form:

$$\begin{array}{rcl} \dot{z} &=& Nz + P\xi \\ \dot{\xi}_1^i &=& \xi_2^i \\ &\vdots& \\ \dot{\xi}_{r_i-1}^i &=& \xi_{r_i}^i \\ \dot{\xi}_{r_i}^i &=& R_i z + S_i \xi + c_i A^{r_i-1} B u \\ y_i &=& \xi_1^i, \ i = 1, \ldots, m. \end{array} \tag{5.3}$$

Then the control:

$$u = Fx + Gv = L^{-1}(-Rz - S\xi + v)$$

where $L$ is given in (4.7) and $R$ and $S$ are defined in (4.9), solves the noninteracting control problem.

The necessity can be shown as follows. If the noninteracting control problem is solved, then there exists $(\bar{r}_1, \ldots, \bar{r}_m)$ such that

$$c_i(A + BF)^k (BG)_j = 0, \ \forall k \geq 0 \text{ where } i \neq j,$$

and for $i = 1, \ldots, m$,

$$\begin{array}{l} c_i(A + BF)^k (BG)_i = 0, \ k = 0, \ldots, \bar{r}_i - 2, \\ c_i(A + BF)^{\bar{r}_i - 1} (BG)_i \neq 0. \end{array}$$

One can easily derive from the above conditions that for $i = 1, \ldots, m$

$$c_i A^k b_j = 0, \ \forall k = 0, 1, \ldots, \bar{r}_i - 2, \ j = 1, \ldots, m$$

and

$$c_i A^{\bar{r}_i - 1} (BG)_j = 0, \ \forall j \neq i, \ c_i A^{\bar{r}_i - 1} (BG)_i \neq 0.$$

Namely,

$$LG := \begin{pmatrix} c_1 A^{\bar{r}_1 - 1} B \\ \vdots \\ c_m A^{\bar{r}_m - 1} B \end{pmatrix} G$$

is nonsingular. Thus it is necessary that $L$ is nonsingular. Therefore the system has relative degree $(\bar{r}_1, \ldots, \bar{r}_m)$. ∎

EXAMPLE 5.1 (Robust car steering). *Let us revisit Example 1.2 in the introduction:*

$$\begin{array}{rcl} \dot{\alpha}_f &=& a_{11}\alpha_f + r + \dot{\delta}_f \\ \dot{\psi} &=& r \\ \dot{r} &=& a_{21}\alpha_f + a_{22} r + b_{21}\delta_f + d(t) \\ y_1 &=& \alpha_f \\ y_2 &=& \psi \end{array}$$

*If we treat $\dot{\delta}_f$ and $\delta_f$ as two controls, it is easy to see that the noninteracting control is solvable since the system would have relative degree* $(1, 2)$*.*

*In particular, the following robust steering law*

$$\dot{\delta}_c = r_{ref} - r = i(v)\delta_s - r$$

*where $\delta_f = \delta_c + \delta_s$ and $\delta_s$ denotes driver input, has been introduced in the literature. $i(v)$ can be considered as the driving habit of an intelligent driver in terms of controlling the yaw motion. This controller has been tested successfully in a modified luxury sedan for compensating unexpected yaw motions, but was not introduced commercially for various reasons.*

So far we have shown that systems having a relative degree can be rendered noninteractive via state feedbacks. We should point out that it is possible, under some assumptions, to expand a system without a relative degree, into one with. Naturally, this has to be done by a dynamic state feedback. Here we will use only an example to illustrate the idea.

EXAMPLE 5.2 (Noninteracting control by dynamic feedback). *Consider*

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_4 + u_1 \\ \dot{x}_3 &= -x_3 + x_4 \\ \dot{x}_4 &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned}$$

*It is easy to calculate that*

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

*Therefore the system does not have any relative degree, and thus noninteracting control by static feedback is not possible. The reason why the system has no relative degree is that the lowest derivatives of $y_1$ and $y_2$ that are affected directly by the input are affected both by $u_1$ and none by $u_2$. Therefore we can insert an "integrator" before $u_1$ so the effect of the first control channel on the output can be "delayed":*

$$\begin{aligned} \dot{\xi} &= v_1 \\ u_1 &= \xi \end{aligned}$$

*Now we consider $\xi$ as a new state and $v_1$ as a control. For consistency of notation we let*

$$v_2 = u_2.$$

*Then for the augmented system with $v_1$ and $v_2$ as control, we have*

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

*Thus noninteracting control is possible.*

### 5.2. Tracking with stability

Consider again system (5.1):

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{5.4}$$

As usual, we suppose the system is controllable and observable. The problem we study in the section is that what kind of reference signals $y_d \in R^m$ can be tracked by the system while some stability is guaranteed.

DEFINITION 5.1. *A bounded reference signal $y_d(t)$ can be tracked asymptotically with stability, if there exists a control $u(t) = Fx(t) + D(t)$ such that $y(t) - y_d(t)$ tends to zero as $t \to \infty$ and $x(t)$ stay bounded.*

THEOREM 5.2. *Suppose system (5.4) has relative degree $(r_1, \ldots, r_m)$ and asymptotically stable zero dynamics. Then all reference signals $y_d(t)$, such that for each $i = 1, \ldots, m$, $y_d^{i(j)}$ $j = 0, \ldots, r_i - 1$ are bounded, can be asymptotically tracked with stability.*

PROOF
Define tracking errors $e_j^i := c_i A^{j-1} x - y_d^{i(j-1)}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, r_i$. Then we can rewrite the system as

$$\begin{aligned} \dot{z} &= Nz + Pe + PY_d \\ \dot{e}_1^i &= e_2^i \\ &\vdots \\ \dot{e}_{r_i-1}^i &= e_{r_i}^i \\ \dot{e}_{r_i}^i &= R_i z + S_i e + S_i Y_d + y_d^{i(r_i)} + c_i A^{r_i-1} Bu \\ y_i &= \xi_1^i, \ i = 1, \ldots, m. \end{aligned} \tag{5.5}$$

$R$ and $S$ are defined in (4.9),

$$e = (e_1^1, \ldots, e_1^m, \ldots, e_{r_m}^m)^T \ Y_d = (y_d^1, \ldots, y_d^m, \ldots, y_d^{m(r_m-1)})^T.$$

As we discussed before, since $L$ is nonsingular, we can find a feedback transformation to convert the last differential equation in (5.5) into

$$\dot{e}_{r_i}^i = v, \ i = 1, \ldots, m.$$

Then it is easy to see that we can design a $v$ to stabilize the error equations. Since the zero dynamics is stable and $Y_d$ is bounded, the result is proven. ∎

REMARK 5.2. *In this setting, we need to include $y_d^{j(i)}$ $i = 0, \ldots, r_j - 1$ in the controller. In Chapter 7 we will take a different approach, where we assume that we know the model that generates the reference (or disturbance) signals, thus less measurements will be needed for the controller.*