Mathematical Systems Theory: Advanced Course
Exercise Session 2

1 Reachability subspace

Suppose that $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are given.

- A subspace $\mathcal{R}$ is a *reachability subspace* if there exist matrices F and G such that
$$\mathcal{R} = \langle A + BF | \text{Im } BG \rangle .$$
- How can we check if a given (A, B)-invariant subspace $\mathcal{R}$ is a reachability subspace? (See Corollary 2.6 in page 15 in the lecture note.)

  Check if the following holds:
$$\mathcal{R} = \langle A + BF | \text{Im } B \cap \mathcal{R} \rangle$$
  where F is an arbitrary friend of $\mathcal{R}$.
- Suppose that $\mathcal{R}$ is a reachability subspace. How can we construct G? (To obtain a friend F of $\mathcal{R}$, see the note for Exercise Session 1.)

  We find $G \in R^{m\times m}$ that satisfies
$$\text{Im } B \cap \mathcal{R} = \text{Im } BG.$$
  Suppose that Im $B \cap \mathcal{R}$ is a subspace spanned by linearly independent column vectors $p_1, \cdots, p_r$ ($p_i \in R^n$). Then, we can obtain linearly independent vectors $u_1, \cdots, u_r$ ($u_i \in R^m$) such that
$$\begin{bmatrix} p_1 & \cdots & p_r \end{bmatrix} = B \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}.$$
  Choose $u_{r+1}, \cdots, u_m$ so that $\{u_i\}_{i=1}^m$ is a basis for R^m. If we take
$$G := \begin{bmatrix} u_1 & \cdots & u_r & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}^{-1},$$
  then
$$BG \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} = \begin{bmatrix} p_1 & \cdots & p_r & 0 & \cdots & 0 \end{bmatrix}$$
  and hence Im $B \cap \mathcal{R} =$ Im BG holds.

Note. If Im $B \cap \mathcal{R}$ is spanned by a subset of columns of B, then it is VERY EASY to construct G satisfying Im $B \cap \mathcal{R}$ = Im BG. Suppose that Im $B \cap \mathcal{R}$ becomes a span of some subset of $\{b_j\}_{j=1}^m$. If Im $B \cap \mathcal{R} = \text{span}\left\{b_{k_1}, \cdots, b_{k_p}\right\}$, then we choose G as a diagonal matrix with one at (k_j, k_j)-elements for $j = 1, \ldots, p$ and with zero at other elements.

- How can we construct the maximal reachability subspace $\mathcal{R}^*$ contained in a given subspace $\mathcal{Z}$? (See Theorem 2.8 in page 15 in the lecture note.)

$$\mathcal{R}^* = \langle A + BF | \text{Im } B \cap \mathcal{S}^*(\mathcal{Z}) \rangle ,$$

$$\begin{array}{cl} \mathcal{S}^*(\mathcal{Z}) & : \text{maximal } (A, B)\text{-invariant subspace in } \mathcal{Z}, \\ F & : \text{a friend of } \mathcal{S}^*. \end{array}$$

Hence, to obtain $\mathcal{R}^*$, we need $\mathcal{S}^*(\mathcal{Z})$. In the next section, we consider $\mathcal{Z} = \ker C$ (which is typical in control problems in this course), and explain the procedure to derive $\mathcal{V}^* := \mathcal{S}^*(\ker C)$.

Problem (Reachability subspace)

Suppose that

$$A := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Is $\mathcal{S} := \text{span}\{e_2\}$ (A, B)-invariant? Is $\mathcal{S}$ a reachability subspace?

2 Computing $\mathcal{V}^*$

Given matrices A, B and C, the maximal (A, B)-invariant subspace in $\ker C$, denoted by $\mathcal{V}^*$, can be obtained by two procedures.

Method 1: $\mathcal{V}^*$-algorithm

Step 0: Form a matrix V_0 whose columns are a basis of $\ker C$. Set $i = 0$.

Step 1: Obtain a matrix Z_i, with the maximal number of linearly independent row vectors, satisfying

$$Z_i \begin{bmatrix} V_i & B \end{bmatrix} = 0$$

Step 2: Obtain a matrix V_{i+1}, with the maximal number of column vectors, satisfying

$$\begin{bmatrix} C \\ Z_i A \end{bmatrix} V_{i+1} = 0$$

Step 3: If the two subspaces $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$, spanned by the columns V_i and V_{i+1} respectively, coincide, then stop. (Note that it may happen that V_i and V_{i+1} are different but $\mathcal{V}_i = \mathcal{V}_{i+1}$) Denoting the columns by $\{v_j\}_{j=1}^{p}$,

$$\mathcal{V}^* = \text{span}\{v_1, \cdots, v_p\}.$$

Otherwise, increment i by one and go back to Step 1.

Note that this algorithm will converge in a finite step, due to Theorem 3.3 in page 23 in the lecture note.

Method 2: Ω^*-algorithm

Denote $G = ImB$.

Step 0: $\Omega_0 = Span\{C\}$,

Step k: $\Omega_k = \Omega_{k-1} + L_{Ax}(\Omega_{k-1} \cap G^{\perp})$. Where $L_{Ax}(\Omega_{k-1} \cap G^{\perp})$ is the span of all row vectors ωA where $\omega \in \Omega_{k-1} \cap G^{\perp}$.

If there is a k^* such that $\Omega_{k^*+1} = \Omega_{k^*}$, then

$$\mathcal{V}^* = \Omega_{k^*}^{\perp}.$$

Example

For the following (A, B, C), compute the maximal (A, B)-invariant subspace in $\ker C$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Method 1: $\mathcal{V}^*$-algorithm

Step 0: First, compute $\ker C$.

$$\begin{aligned}\ker C &= \{x \in R^3 : Cx = 0\} \\ &= \{x \in R^3 : x_1 + x_2 + x_3 = 0\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} : \; x_1 \in R, \; x_2 \in R \right\} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} =: \mathcal{V}_0.\end{aligned}$$

Therefore,

$$V_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Step 1: Solve $Z_0 \begin{bmatrix} V_0 & B \end{bmatrix} = 0$ for Z_0.

$$Z_0 \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & -1 & -2 & -1 \end{bmatrix} = 0 \implies Z_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Step 2: Solve $\begin{bmatrix} C \\ Z_0 A \end{bmatrix} V_1 = 0$ for V_1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} V_1 = 0 \implies V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Step 3: Since $\mathcal{V}_1 := \text{span}\{V_1\}$ is different from $\mathcal{V}_0$, go back to Step 1.

Step 1-2: Solve $Z_1 \begin{bmatrix} V_1 & B \end{bmatrix} = 0$ for Z_1.

$$Z_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & -2 \end{bmatrix} = 0 \implies Z_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Step 2-2: Solve $\begin{bmatrix} C \\ Z_1 A \end{bmatrix} V_2 = 0$ for V_2. Then, $V_2 = V_1$.

Step 3-2: Since $\mathcal{V}_2 := \text{span}\{V_2\}$ equals to $\mathcal{V}_1$,

$$\mathcal{V}^* = \mathcal{V}_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}.$$

Method 2: Ω^*-algorithm

$$G = ImB = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix} \Rightarrow G^\perp = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Step 0: $\Omega_0 = Span\{C\} = Span\{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\}$,

Step 1: $\Omega_1 = \Omega_0 + L_{Ax}(\Omega_0 \cap G^\perp)$.

$\Omega_0 \cap G^\perp = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = w$

$wA = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}, \Rightarrow L_{Ax}(\Omega_0 \cap G^\perp) = Span\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$

So $\Omega_1 = \Omega_0 + L_{Ax}(\Omega_0 \cap G^\perp) = Span\left\{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}\right\}$

Step 2: $\Omega_1 \cap G^\perp = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = w$, Therefore, we have $\Omega_2 = \Omega_1$, so $\Omega^* = \Omega_1$

Then $\mathcal{V}^*$ is computed as $\mathcal{V}^* = \Omega^{*\perp} = Span\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$.

What is the maximal reachability subspace $\mathcal{R}^*$ in this example? To compute $\mathcal{R}^*$, we need a friend F of $\mathcal{V}^*$. Since

$$A\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{V} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{V}(\underbrace{-1}_{K}) + \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -2 & -1 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{U},$$

by solving $FV = -U$, we obtain a solution

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{R}^* &= \langle A + BF|\text{Im } B \cap \mathcal{V}^*\rangle \\ &= \left\langle \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \middle| \text{Im} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\rangle = \text{Im}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}. \end{aligned}$$

From the definition of a reachability subspace, there is a G satisfying

$$\mathcal{R}^* = \langle A + BF | \text{Im } BG \rangle .$$

How can we obtain G? We aim at choosing G with

$$\text{Im } B \cap \mathcal{R}^* = \text{Im } BG.$$

We achieve this relation with $G = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Problem (Finding $\mathcal{V}^*$)

For the following (A, B, C), compute $\mathcal{V}^*$.

1. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$

2. $A = \begin{bmatrix} 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$

3. $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$

3 Relative degree and normal form

Relative degree (Square MIMO case)

Suppose that (A, B, C) is minimal and that B and C have linearly independent columns and rows, respectively. System

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} x, \; c_i \in R^{1 \times n} \end{cases}$$

with m-inputs ($u \in R^m$) and m-outputs ($y \in R^m$) has *relative degree* $(r_1, \cdots, r_m)$ if for $i = 1, \ldots, m$,

$$\begin{array}{ll} c_i A^j B = 0_{1\times m}, & j = 0, 1, \ldots, r_i - 2 \\ c_i A^{r_i - 1} B \neq 0_{1\times m}, & \end{array}$$

and the matrix

$$L := \begin{bmatrix} c_1 A^{r_1 - 1} B \\ \vdots \\ c_m A^{r_m - 1} B \end{bmatrix}$$

is nonsingular.

Example (Relative degree)

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

For c_1,

$$\begin{array}{l} c_1 B = \begin{bmatrix} 0 & 0 \end{bmatrix} \\ c_1 AB = \begin{bmatrix} 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix} \implies r_1 = 2 \end{array}$$

For c_2,

$$c_2 B = \begin{bmatrix} 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix} \implies r_2 = 1$$

The matrix L becomes

$$L := \begin{bmatrix} c_1 AB \\ c_2 B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

and it is clear that L is nonsingular. Hence, the system has relative degree

$$(r_1, r_2) = (2, 1) .$$

Normal form

Once we have obtained relative degree, we can change coordinates of the system to transform it into a normal form.

$$
\begin{cases}
\dot{z} &= Nz + P\xi \\
\dot{\xi}_1^i &= \xi_2^i \\
\dot{\xi}_2^i &= \xi_3^i \\
 &\vdots \\
\dot{\xi}_{r_i-1}^i &= \xi_{r_i}^i \\
\dot{\xi}_{r_i}^i &= R_i z + S_i \xi + c_i A^{r_i-1} B u \\
y_i &= \xi_1^i, \quad i = 1, \dots, m.
\end{cases}
$$

Normal form is useful in obtaining zero dynamics and in solving several control problems (see Chapter 5 in the lecture note).

First, choose the new states as follows.

$$
\left.
\begin{array}{l}
\begin{cases}
\xi_1^1 &:= c_1 x \\
\xi_2^1 &:= c_1 A x \\
 &\vdots \\
\xi_{r_1}^1 &:= c_1 A^{r_1-1} x
\end{cases} \\
\begin{cases}
\xi_1^2 &:= c_2 x \\
\xi_2^2 &:= c_2 A x \\
 &\vdots \\
\xi_{r_2}^2 &:= c_2 A^{r_2-1} x
\end{cases} \\
\qquad\vdots \\
\begin{cases}
\xi_1^m &:= c_m x \\
\xi_2^m &:= c_m A x \\
 &\vdots \\
\xi_{r_m}^m &:= c_m A^{r_m-1} x
\end{cases}
\end{array}
\right.
\Longrightarrow \xi := \begin{bmatrix} \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \\ \vdots \\ \xi_{r_1}^1 \end{bmatrix} \\ \begin{bmatrix} \xi_1^2 \\ \xi_2^2 \\ \vdots \\ \xi_{r_2}^2 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \xi_1^m \\ \xi_2^m \\ \vdots \\ \xi_{r_m}^m \end{bmatrix} \end{bmatrix} = \underbrace{\begin{bmatrix} \begin{bmatrix} c_1 \\ c_1 A \\ \vdots \\ c_1 A^{r_1-1} \end{bmatrix} \\ \begin{bmatrix} c_2 \\ c_2 A \\ \vdots \\ c_2 A^{r_2-1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} c_m \\ c_m A \\ \vdots \\ c_m A^{r_m-1} \end{bmatrix} \end{bmatrix}}_{=:T_\xi} x
$$

Note that $\xi \in R^{(r_1+\cdots+r_m)}$. Since $x \in R^n$, for the coordinate change, we need to add another $n - (r_1 + \cdots + r_m)$ states. We choose these states as

$$z := T_z x,$$

where T_z is a matrix of size $(n - (r_1 + \cdots + r_m)) \times n$ and satisfies

- $T_z B = 0$
- $T := \begin{bmatrix} T_z \\ T_\xi \end{bmatrix}$ is nonsingular.

Why is such a choice of T_z possible? Since the columns of B span m dimensional subspace Im B in R^n, there is an $(n-m)$ dimensional subspace $\mathcal{W}$ which is orthogonal to Im B, i.e.,

$$R^n = \text{Im } B + \mathcal{W}, \ \mathcal{W} \perp \text{Im } B.$$

In T_ξ, there are $((r_1 - 1) + \cdots + (r_m - 1)) = (r_1 + \cdots + r_m - m)$ linearly independent row vectors in $\mathcal{W}$. Therefore, we can choose another

$$n - m - (r_1 + \cdots + r_m - m) = n - (r_1 + \cdots + r_m)$$

linearly independent row vectors in $\mathcal{W}$.

The new state vector is

$$\begin{bmatrix} z \\ \xi \end{bmatrix} = \underbrace{\begin{bmatrix} T_z \\ T_\xi \end{bmatrix}}_{=:T} x.$$

Therefore,

$$\begin{aligned}
\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} &= T\dot{x} \\
&= T(Ax + Bu) \\
&= TAT^{-1} \begin{bmatrix} z \\ \xi \end{bmatrix} + TBu \\
&= TAT^{-1} \begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ T_\xi B \end{bmatrix} u \quad (\text{since } T_z B = 0), \\
y &= Cx = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} x = \begin{bmatrix} \xi_1^1 \\ \vdots \\ \xi_1^m \end{bmatrix}.
\end{aligned}$$

Here, TAT^{-1} and $T_\xi B$ have special structures.

Example (Normal form)

Consider the same system as before, i.e.,

$$A = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Relative degree is $(2,1)$. So, we choose the new states as

$$\xi := \begin{bmatrix} \xi_1^1 \\ \xi_2^1 \\ \xi_1^2 \end{bmatrix} := \begin{bmatrix} c_1 \\ c_1 A \\ c_2 \end{bmatrix} x = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} x.$$

By adding another state $z := \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$,

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ -9 & 7 & -3 & 1 \\ -5 & 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix}. \end{aligned}$$

or equivalently,

$$\begin{aligned} \dot{z} &= \underbrace{1}_{N} \cdot z + \underbrace{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}}_{P} \xi \\ \dot{\xi}_1^1 &= \xi_2^1 \\ \dot{\xi}_2^1 &= \underbrace{-9}_{R_1} z + \underbrace{\begin{bmatrix} 7 & -3 & 1 \end{bmatrix}}_{S_1} \xi + \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{c_1 AB} u \\ y_1 &= \xi_1^1 \\ \dot{\xi}_1^2 &= \underbrace{-5}_{R_2} z + \underbrace{\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}}_{S_2} \xi + \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{c_2 B} u \\ y_2 &= \xi_1^2 \end{aligned}$$