Mathematical Systems Theory: Advanced Course
Exercise Session 6

1 Normal form in SISO nonlinear systems

Consider a SISO nonlinear system

$$\begin{cases} \dot{x} & = & f(x) + g(x)u \\ y & = & h(x). \end{cases}$$

The system has *relative degree at a point* x_0 if

$$\begin{aligned} & L_g L_f^k h(x) = 0, \forall x \in \mathcal{N}(x_0),\ \ k = 0, 1, \dots, r-2, \\ & L_g L_f^{r-1} h(x_0) \neq 0. \end{aligned}$$

If the system has relative degree at x_0, then in $\mathcal{N}(x_0)$, we can transform the system into a normal form:

$$\begin{cases} \dot{z} & = & f_0(z,\xi), \\ \dot{\xi}_1 & = & \xi_2 \\ & \vdots & \\ \dot{\xi}_{r-1} & = & \xi_r \\ \dot{\xi}_r & = & f_1(z,\xi) + g_1(z,\xi)u. \end{cases}$$

The zero dynamics is

$$\dot{z} = f_0(z, 0).$$

To obtain a normal form, we take new states as

$$\xi_1 := h(x), \xi_2 := L_f h(x), \cdots, \xi_r := L_f^{r-1} h(x).$$

As for the z part, first define

$$\mathcal{D} := \operatorname{span}\{g\}.$$

Then, compute

$$\mathcal{D}^\perp := \{w_i(x) :\ i = 1, \dots, n-1,\ w_i(x)g = 0\}.$$

For each row vector $w_i(x) =: \begin{bmatrix} w_1^i(x) & \cdots & w_n^i(x) \end{bmatrix}$, if the following holds:

$$\frac{\partial w_j^i}{\partial x_k} = \frac{\partial w_k^i}{\partial x_j},\ \forall j, k,$$

then you can find z_i satisfying

$$dz_i = w_i.$$

Choose such z_i that are linearly independent of ξ part that has already been chosen.

Otherwise, you have to change the basis of $\mathcal{D}^\perp$. (But how to find such basis is not required in this course.)

Example

Consider the system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 + u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \sin 2x_1 + (\cos x_1)u \\ y &= x_1, \end{cases}$$

or equivalently,

$$\begin{cases} \dot{x} &= \underbrace{\begin{bmatrix} x_2 \\ \sin x_1 \\ x_4 \\ \sin 2x_1 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \cos x_1 \end{bmatrix}}_{g(x)} u \\ y &= \underbrace{x_1}_{h(x)} \end{cases}$$

First, let us check if the system has relative degree at $x = 0$.

$$\begin{aligned} L_g h(x) &= \frac{\partial h}{\partial x} g = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} g = 0 \\ L_g L_f h(x) &= L_g \left(\frac{\partial h}{\partial x} f \right) = L_g(x_2) = \frac{\partial x_2}{\partial x} g = 1 \neq 0. \end{aligned}$$

Hence, relative degree is two.

Next, we transform the system into a normal form. We take new states as

$$\xi_1 := h(x) = x_1, \ \xi_2 := L_f h(x) = x_2.$$

We have to take another two states z_1 and z_2 (z part). To this end, we first find

$$\mathcal{D}^\perp := (\text{span}\,\{g\})^\perp = \text{span}\left\{ e_1^T, e_3^T, \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix} \right\}.$$

We obtain one state z_1 from the following observation:

$$
\begin{aligned}
&dz = e_1^T \Rightarrow z_1 = x_1 \text{ (already chosen as } \xi_1\text{. Ignore!)}\\
&dz = e_3^T \Rightarrow z_1 = x_3.
\end{aligned}
$$

To ensure the existence of z_2 with $dz_2 = \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix}$, we verify

$$\frac{\partial \cos x_1}{\partial x_4} = \frac{\partial(-1)}{\partial x_2} (= 0).$$

So we can solve

$$dz_2 = \begin{bmatrix} * & \cos x_1 & * & -1 \end{bmatrix}.$$

or equivalently,

$$\begin{cases} \dfrac{\partial z_2}{\partial x_2} &= \cos x_1 \\ \dfrac{\partial z_2}{\partial x_4} &= -1 \end{cases}$$

One solution is

$$z_2 = (\cos x_1)x_2 - x_4.$$

Since $\xi_1 := x_1$, $\xi_2 := x_2$ and $z_1 := x_3$ do not include x_4, this z_2 satisfies the second condition above.

Therefore,

$$
\begin{aligned}
\dot{z}_1 &= \dot{x}_3 = x_4 = (\cos x_1)x_2 - z_2 = (\cos \xi_1)\xi_2 - z_2 \\
\dot{z}_2 &= (-\sin x_1)\dot{x}_1 x_2 + (\cos x_1)\dot{x}_2 - \dot{x}_4 \\
&= -(\sin x_1)x_2^2 + (\cos x_1)(\sin x_1 + u) - (\sin 2x_1 + (\cos x_1)u) \\
&= -(\sin \xi_1)\xi_2^2 - \frac{1}{2}\sin 2\xi_1 \\
\dot{\xi}_1 &= \dot{x}_1 = x_2 = \xi_2 \\
\dot{\xi}_2 &= \dot{x}_2 = \sin x_1 + u = \sin \xi_1 + u \\
y &= \xi_1.
\end{aligned}
$$

The zero dynamics is obtained by setting $\xi = 0$:

$$
\begin{aligned}
\dot{z}_1 &= -z_2 \\
\dot{z}_2 &= 0.
\end{aligned}
$$

2 Local feedback stabilization

Consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u.$$

To check the local stabilizability of this system, follow the procedure below.

1. First, you should **always** check if the linearized system with

$$A := \frac{\partial f}{\partial x}(0), \;\; b = g(0)$$

is controllable (or stabilizable). If it is, then the nonlinear system is locally stabilizable.

2. If Step 1 fails, then use Proposition 8.23 (page 77) in case the system can be transformed into a normal form:

$$\begin{aligned} \dot{z} &= f_0(z, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ &\vdots \\ \dot{\xi}_{r-1} &= \xi_r \\ \dot{\xi}_r &= f_1(z, \xi) + g_1(z, \xi)u \\ y &= \xi_1. \end{aligned}$$

If the zero dynamics of the system is locally asymptotically stable, then the stabilizing control is

$$u = \frac{1}{g_1(z, \xi)}(-f_1(z, \xi) - a_r\xi_1 + \cdots - a_1\xi_r),$$

where a_i, $i = 1, \ldots, r$ are chosen so that the polynomial

$$s^r + a_1 s^{r-1} + \cdots + a_r$$

becomes Hurwitz polynomial (i.e., all the roots are in the open left half-plane.)

3 Exact linearization

Consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u, \;\; x \in \mathcal{N}(x^0) \subset R^n$$

We want to find

- a feedback $u = \alpha(x) + \beta(x)v$, and
- a coordinate change $z = \phi(x)$,

so that the resulting system becomes a linear system:

$$\dot{z} = Az + bv,$$

where (A, b) is controllable.

Proposition 8.20

The exact linearization problem is solvable at x^0 if and only if

1. rank $\left[\begin{array}{cccc} g(x^0) & ad_f g(x^0) & \cdots & ad_f^{n-1} g(x^0) \end{array}\right] = n$
2. The distribution $\mathcal{D}(x) := \text{span}\left\{g(x), ad_f g(x), \cdots, ad_f^{n-2} g(x)\right\}$ is involutive in $N(x^0)$.

Here,

$$ad_f^0 g := g,\ ad_f^1 g := [f, g]\,,\ ad_f^{k+1} g := \left[f, ad_f^k g\right],$$

and $\mathcal{D}$ is *involutive* if for any $k_1, k_2 \in \mathcal{D}$,

$$[k_1, k_2] \in \mathcal{D}.$$

The method to obtain a feedback $u = \alpha(x) + \beta(x)v$ and a coordinate change $z = \phi(x)$ is explained through an example.

Example

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_3 \sin^2 x_1 + u \\ \dot{x}_2 &= 2x_3 \cos^2 x_1 - 2u \\ \dot{x}_3 &= 2 \sin x_2, \end{aligned}$$

namely,

$$\dot{x} = \underbrace{\begin{bmatrix} x_3 \sin^2 x_1 \\ 2x_3 \cos^2 x_1 \\ 2 \sin x_2 \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{g} u$$

First, using Proposition 8.20, we check the solvability of the exact linearization at $x = 0$.

1. $ad_f g(0)$ and $ad_f^2 g(0)$ are computed as

$$
\begin{aligned}
ad_f g(0) &= [f, g]_{x=0} = \cdots = \begin{bmatrix} -x_3 \sin 2x_1 \\ 2x_3 \sin 2x_1 \\ 2\cos x_2 \end{bmatrix}_{x=0} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\
ad_f^2 g(0) &= [f, ad_f g]_{x=0} = \cdots = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\operatorname{rank} \begin{bmatrix} g(0) & ad_f g(0) & ad_f^2 g(0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix} = 3.
$$

2. Check if the distribution $\mathcal{D} := \operatorname{span}\{g, ad_f g\}$ is involutive in $\mathcal{N}(0)$.

$$
\begin{aligned}
[g, ad_f g] &= \frac{\partial ad_f g}{\partial x} g - \frac{\partial g}{\partial x} ad_f g = \begin{bmatrix} -2x_3 \cos 2x_1 \\ 4x_3 \cos 2x_1 \\ 4 \sin x_2 \end{bmatrix}. \\
&= 2x_3(\tan x_2 \sin 2x_1 - \cos 2x_1) \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{g} + 2\tan x_2 \underbrace{\begin{bmatrix} -x_3 \sin 2x_1 \\ 2x_3 \sin 2x_1 \\ 2\cos x_2 \end{bmatrix}}_{ad_f g} \\
&\in \mathcal{D}.
\end{aligned}
$$

Hence $\mathcal{D}$ is involutive in $\mathcal{N}(0)$.

We want to find $\lambda(x)$ such that the system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
y &= \lambda(x)
\end{aligned}
$$

has relative degree three. Such λ is obtained by finding $\mathcal{D}^\perp$:

$$
\mathcal{D}^\perp = \operatorname{span}\{w\} = \operatorname{span}\left\{\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}\right\}.
$$

In this case, since w is a constant vector, there exists a λ satisfying

$$
d\lambda = w.
$$

Such λ can be easily found by inspection.

$$\lambda = 2x_1 + x_2.$$

With the obtained λ, the system has relative degree three. Hence, by doing a coordinate change as

$$\begin{aligned} \xi_1 &:= \lambda(x) = 2x_1 + x_2 \\ \xi_2 &:= L_f\lambda(x) = 4x_3 \\ \xi_3 &:= 4\sin x_2, \end{aligned}$$

we can transform the system into a normal form:

$$\begin{cases} \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \dot{\xi}_3 = L_f^3\lambda + L_gL_f^2\lambda u \\ y = \xi_1. \end{cases}$$

Thus, the exact linearization can be achieved by the feedback

$$u = -\frac{L_f^3\lambda}{L_gL_f^2\lambda} + v,$$

and the coordinate change above.

3.1 Multi-agent consensus

Consider N agents

$$\dot{x}_i = u_i, \quad i = 1, \cdots, N.$$

Suppose each agent uses the following neighborhood control:

$$u_i = \sum_{j \in N_i} (x_j - x_i),$$

where N_i indicates the neighbors of agent i.

We say the consensus is reached if as $t \to \infty$ we have

$$x_1(t) = x_2(t) = \cdots = x_N(t).$$

Solvability condition(Proposition 9.2)

The consensus problem is solved if the associated neighborhood graph is connected.

Example

We consider a three-agent system:

$$\dot{x}_i = u_i,\ \ i = 1, 2, 3.$$

Case 1: $N_1 = 2, N_2 = \{1, 3\}, N_3 = 2$. Then

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 &= x_2 - x_3.\end{aligned}$$

Let $\bar{x} = Px$, where

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$\bar{A} = PAP^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly, A has one eigenvalue at zero and two eigenvalues at -2.

Case 2: $N_1 = \{2, 3\}, N_2 = \{1, 3\}, N_3 = \{1, 2\}$. Then

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 + x_3 - x_1 \\ \dot{x}_2 &= x_1 - x_2 + x_3 - x_2 \\ \dot{x}_3 &= x_1 - x_3 + x_2 - x_3.\end{aligned}$$

Once again we let $\bar{x} = Px$, then

$$\bar{A} = PAP^{-1} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$

In this case A has one eigenvalue at zero and two eigenvalues at -3. This suggests that with more information available, the agents reach consensus faster.