SF2842: Geometric Control Theory, 2014
Answers of Homework 2

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1. **Solution**:

a. $x_1 = 0 \;\Rightarrow\; \dot{x}_1 = x_1 + ax_2 = 0 \;\Rightarrow\; ax_2 = 0.$
If $a = 0$, then we have $cA^k b = 0$ for any $k \in \mathbb{Z}^+$. Thus, the system does not have relative degree.
<u>If $a \neq 0$, then we have $x_2 = 0 \;\Rightarrow\; \dot{x}_2 = 2x_2 + ax_3 + u = 0 \;\Rightarrow\; u = -ax_3$. The system has relative degree 2.</u>

b. For $a \neq 0$, let $\xi_1 = x_1$, $\xi_2 = x_2$, $z_1 = x_3$ and $z_2 = x_2 - x_4$.
$\dot{z}_1 = \dot{x}_3 = x_1 + x_4 = -z_2 + \xi_1 + \xi_2$
$\dot{z}_2 = \dot{x}_2 - \dot{x}_4 = 2x_2 + ax_3 - x_2 + x_4 = az_1 - z_2 + 2\xi_2$
<u>Zero dynamics: $\dot{z} = \begin{pmatrix} 0 & -1 \\ a & -1 \end{pmatrix} z$</u>

c. If $a = -2$, the matrix $\begin{pmatrix} 0 & -1 \\ a & -1 \end{pmatrix}$ has eigenvalues $1, -2$.
$P(s) = \det \begin{pmatrix} sI - A & b \\ -c & 0 \end{pmatrix} = \cdots = -2s^2 - 2s + 4 = 0 \;\Rightarrow\; \underline{s = 1, -2}.$

2. **Solution**:

a. $A\Pi - \Pi\Gamma = -bq$
$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} - \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\Rightarrow\; \underline{\Pi = \begin{pmatrix} 1 & -\frac{4}{9} \\ 0 & \frac{1}{9} \end{pmatrix}}.$$

b. If $a = 1$, the pair (c, A) becomes unobservable. If $a = 0$ or -2, at least one of the eigenvalues of the matrix Γ will be the transmission zero of the system (A, b, c).
Hence, <u>if $a \neq -2, 1, (0)$, the big system is observable.</u>

3. **Solution**:

The system can be written as

$$
\begin{aligned}
\begin{pmatrix}\dot{\alpha}_f \\ \dot{\psi} \\ \dot{r}\end{pmatrix} &= \begin{pmatrix}-2 & 0 & -0.5 \\ 0 & 0 & 1 \\ -0.6 & -3.5 & 0\end{pmatrix}\begin{pmatrix}\alpha_f \\ \psi \\ r\end{pmatrix} + \begin{pmatrix}0 \\ 0 \\ 1\end{pmatrix} u \\
\begin{pmatrix}\dot{w}_1 \\ \dot{w}_2\end{pmatrix} &= \begin{pmatrix}0 & 2 \\ -2 & 0\end{pmatrix}\begin{pmatrix}w_1 \\ w_2\end{pmatrix} \\
u &= \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}w_1 \\ w_2\end{pmatrix}
\end{aligned}
$$

The solution to the problem can be obtained by solving

$$A\Pi - \Pi\Gamma = -bq$$

$$c\Pi = q$$

while constraining that $c_1 = 0$.

4. **Solution**:

a. FIORP is solvable iff no eigenvalues of S is the transmission zero of the system (A, B, C).
S has eigenvalues 0, $\pm i$.
$P(s) = \det \begin{pmatrix} sI - A & b \\ -c & 0 \end{pmatrix} = \cdots = 2(\alpha + 1 - s) \Rightarrow \underline{a \neq -1}$.

b. The solution to the Sylvester equation:

$$
\begin{aligned}
\Pi s &= A\Pi + B\Gamma + P \\
0 &= C\Pi - Q
\end{aligned}
$$

is

$$\underline{\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; \Gamma = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.}$$