KTH Matematik

SF2842: Geometric Control Theory

Solution to Homework 2

Due February 26, 16:50pm, 2015

You may discuss the problems in group (maximal two students in a group), but each of you **must** write and submit your own report. Write the name of the person you cooperated with.

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1. Consider

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx,\end{aligned}$$

where $x \in R^n,\ u \in R^m,\ y \in R^m$. Show that the maximal controllability (reachability) subspace contained in $ker\ C$ is $\{0\}$ if the system has some relative degree $(r_1, \cdots, r_m)$. [3p]

Answer:

If the system has relative degree, the maximum (A, B)-invariant subspace can be expressed as $V^* = \{x : \quad c_i A^{j-1} x = 0, i = 1, \dots, m$ and $j = 1, \dots, r_i\}$. For any nonzero vector in ImB, $y = Bv \neq 0$, if y is also in V^*, then we have $c_i A^{r_i - 1} B v = 0$ for $i = 1, \dots, m$. If we stack these equations, we will get

$$\begin{pmatrix} c_1 A^{r_1-1} B \\ c_2 A^{r_2-1} B \\ \vdots \\ c_m A^{r_m-1} B \end{pmatrix} v = Lv = 0.$$

By the definition of the relative degree, L is nonsingular, which implies $v = 0$. But this contradicts with the assumption that $y \neq 0$. Hence $V^* \cap \mathrm{Im}B = \{0\}$, and it follows that R^* is $\{0\}$ as well.

2. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ a & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u\\ y &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x,\end{aligned}$$

where a is a constant.

(a) Compute the transmission zeros. [3p]

(b) For what a does the system have relative degree? [2p]

(c) when is the system invertible? [2p]

Answer:

(a) When $a = 0$, the system does not have transmission zero. When $a \neq 0$, $s_0 = \frac{a+1}{a}$ is the transmission zero of the system.

(b) The system has relative degree $(1, 2)$ when $a \neq 0$.

(c) The system is invertible for any a. (Just need to check whether $V^* \cap \mathrm{Im}B = 0$ or not.)

3. Consider a control system subject to disturbance:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 + x_3 + u - w_1 \\
\dot{x}_3 &= -\alpha x_3 - 2u + w_1 \\
y &= x_1,
\end{aligned}$$

where w_1 is an unknown nonzero constant (disturbance), and α a positive constant.

(a) Let $u = 0$ and compute the invariant subspace $x = \Pi w_1$. [2p]

(b) For what value(s) of α is the above system (consisting of x and w_1) unobservable? Explain why. [2p]

(c) Let the desired output $y_d = 1$, for what α is the full information output regulation problem solvable? [3p]

Answer:

(a) One can write the system as $\dot{x} = Ax + Bu + Pw$, $y = Cx$, with appropriate A, B, P, and C. w is a constant implies that $\dot{w} = 0w$. On the invariant subspace $x = \Pi w$, it holds that $\dot{x} = \Pi \dot{w}$, which gives the equation

$$Ax + Bu + Pw = \Pi \cdot 0 \;\Rightarrow\; A\Pi w + Pw = 0,$$

for any w. Solving the equation $A\Pi + P = 0$ results in

$$\Pi = \begin{pmatrix} \frac{1}{\alpha} - 1 \\ 0 \\ \frac{1}{\alpha} \end{pmatrix}.$$

(b) The system is unobservable if and only if 0 is a transmission zero of the system (A, P, C). This happens only when $\alpha = 1$.

(c) The error feedback output regulation problem is solvable if and only if 0 is not a transmission zero of the system (A, B, C). This happens when $\alpha \neq 2$. (Note that when $\alpha = 3$, the system is not controllable, but still stabilizable since A itself is a stable matrix. So it is OK.)

4. Consider the car steering example:

$$\begin{aligned}
\dot{\alpha}_f &= -2\alpha_f + r + 0.5\dot{\delta}_f \\
\dot{\psi} &= r \\
\dot{r} &= -1.5\alpha_f - 1.2\psi + \delta_f + d(t),
\end{aligned}$$

where the driver's goal is to keep the orientation straight ($\delta_f = -0.7\psi$), $d(t)$ is a sinusoidal disturbance $a\sin(2t+\theta)$ with unknown amplitude and phase.

Design an output that is a linear combination of ψ and r, such that the output reconstructs the disturbance in stationarity, and use Matlab simulation to illustrate your result. [3p]

Answer:

The system constrained with $\delta_f = -0.7\psi$ can be written as:

$$\begin{aligned}
\dot{x} &= Ax + bu \\
\dot{w} &= \Gamma w \\
u &= qw,
\end{aligned}$$

where

$$x = \begin{pmatrix} \alpha_f \\ \psi \\ r \end{pmatrix}, A = \begin{pmatrix} -2 & 0 & 0.65 \\ 0 & 0 & 1 \\ -1.5 & -1.9 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \Gamma = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \text{ and } q = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

The output tracking input problem with using only ψ and r can solved by solving

$$\begin{aligned}
A\Pi - \Pi\Gamma &= -bq \\
c\Pi &= q,
\end{aligned}$$

where $c = \begin{pmatrix} 0 & c_2 & c_3 \end{pmatrix}$. It turns out that $c = \begin{pmatrix} 0 & -1.6125 & 0.2438 \end{pmatrix}$ solves the OTIP.