KTH Matematik

Solution to Exam of SF2842 Geometric Control Theory

December 14 2012

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Allowed written material: All course material (except the old exams and their solutions) and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 31 points give grade C, 37 points B and 43 points give grade A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.

Let $S = ker(C)$. Then system (1) is not controllable if $R^* = 0$ (the maximal reachability subspace in S). .... (2p)

Answer: False. Just consider any minimal SISO system for example.

(b) If system (1) is observable, then $V^* = 0$. .... (2p)

Answer: False. Consider a SISO system with finite zeros for example.

(c) Consider an exo-system $\dot{w} = Sw$, where $w \in R^q$. Suppose $q \leq n$, all the eigenvalues of S have non-negative real part, and (q, S) is observable. For system (1) assume (A, B) is controllable and $m = p = 1$. Then we can find C and a feedback control $u = Fx + Hw$ such that $Cx - qw = 0$ as $t \to \infty$.......... (2p)

Answer: True.

(d) Consider a nonlinear system

$$\dot{x} = f(x)$$

where $x \in R^n$ and $f(0) = 0$. Let $A = \frac{\partial f(0)}{\partial x}$. We can always use the principle of stability by first approximation to determine the stability of the nonlinear system if $det(A) \neq 0$. .... (2p)

Answer: False. Consider A has a pair of imaginary eigenvalues for example.

(e) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x), \end{aligned}$$

where $x \in R^n$. If the Lie bracket $[f(x), g(x)] = 0$, then there does not exist any $h(x)$ such that the relative degree of the system is n at any point. .... (2p)
Answer: True.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} u \\ y &= (0\ 0\ 2\ 0)x. \end{aligned}$$

(a) Find V^*. .... (4p)
Answer: $V^* = \{x : x_2 = 0,\ x_3 = 0\}$.

(b) Find R^* in V^*. .... (2p)
Answer: $R^* = V^*$.

(c) Can we find a matrix F, such that $A + BF$ has all the eigenvalues with negative real part and V^* is invariant under $A + BF$? .... (4p)
Answer: Yes.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix} x. \end{aligned}$$

Design coefficients c_1, c_2, c_3, c_4, where each c_i can take value in $\{-1, 0, 1\}$, such that the following two conditions are both satisfied

(a) the noninteracting control problem is solvable, ... (4p)
and

(b) the zero dynamics is asymptotically stable. .... (6p)

Answer: For example, $c = [1\ 0\ 1\ 0]$. The system has rel. degree $\{1, 2\}$ and $\dot{z} = -z$.

4. Consider a control system subject to disturbance:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2 - x_3 + u + w_1 \\
\dot{x}_3 &= -2x_3 - u \\
y &= x_1,
\end{aligned}
$$

where w_1 is an unknown constant (disturbance).

(a) Show that the disturbance decoupling problem (DDP) is not solvable. ... (2p)
Answer: $[0\ 1\ 0]^T$ is not in V^*.

(b) When $u = 0$, show that for any given initial condition, the output converges to a constant as $t \to \infty$. ... (3p)
Answer: Since A is a stable matrix, in steady state $y = c\Pi w = const.$

(c) Show that we can design a dynamical feedback controller that only requires the measurement of y such that for the closed-loop system $y(t)$ converges to zero as $t \to \infty$ (you do not need to design such a controller). ... (5p)
Answer: Since the plant is controllable, the expanded system (with $e = x_1$) is observable, and $s = 0$ is not a transmission zero.

5. Consider in a neighborhood N of the origin

$$
\begin{aligned}
\dot{x}_1 &= \alpha \tan(x_1) + x_2 \tan(x_1) + x_3 + \cos(x_3)u \\
\dot{x}_2 &= -\alpha \sin^2(x_1) - \sin^3(x_1) - x_2 + 2x_3^2 \\
\dot{x}_3 &= x_3 - \cos(x_1)u \\
y &= x_3,
\end{aligned}
$$

where α is a constant.

(a) Convert the system into the normal form (*you need to specify the new coordinates explicitly, but are allowed to use the current variables in the expression of the right hand side of the normal form*). ... (4p)
Answer: $\xi_1 = x_3,\ z_1 = x_2,\ z_2 = \sin x_1 + \sin x_3$.

(b) Analyze the stability of the zero dynamics with respect to the value of α. (4p)
Answer: $\alpha < 0$: exponentially stable; $\alpha \geq 0$: unstable.

(c) Consider the same nonlinear system but without the output. Show that the exact linearization problem is not solvable (*Hint: this does not have to involve a lot of calculations*). ... (2p)
Answer: Since the linearized system is not controllable.