KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

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Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.
If system (1) is observable, then $V^* = 0$. .... (4p)
Answer: False. In the SISO case, $V^* \neq 0$ if there are zeros.

(b) Let $p = m$. If system (1) does not have any relative degree $(r_1, \cdots, r_m)$, then it does not have any (transmission) zero. .... (4p)
Answer: False. We can construct systems such that $dim\ V^* - dim\ R^* > 0$.

(c) Let $p = m$. If system (1) has relative degree $(r_1, \cdots, r_m)$, then the only controllability subspace contained in $ker\ C$ is $\{0\}$........................... (4p)
Answer: True, since $R^* = 0$.

(d) Consider a controllable and observable (when w is set to 0) linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ \dot{w} &= Sw \\ y &= Cx, \end{aligned}$$

where w is disturbance. If $Im\ E$ is not contained in V^*, then the full information output regulation problem (here $y_r = 0$ is the reference output) is never solvable. (4p)
Answer: False, we have counter examples in the compendium.

(e) Consider a nonlinear single-input system

$$\dot{x} = f(x) + g(x)u$$

where $x \in R^n$, $f, g \in C^\infty$ and $f(0) = 0$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$.
A necessary condition for the existence of an output mapping $h(x)$ such that the system has a relative degree $r > 1$ at the origin is that the matrix A is not identically zero. . . . . (4p)

Answer: True. This can be seen from the normal form.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= (1\ 0\ 0\ 0)x. \end{aligned}$$

(a) Find V^*. . . . . (8p)
Answer: $V^* = \{x : x_1 = x_2 = 0\}$.

(b) Can we find a $u(t)$ such that the corresponding trajectory from $x(0) = 0$ to $x(t_1) = (0\ 0\ 1\ 1)^T$ lies completely in V^*? . . . . (6p)
Answer: Yes, since $R^* = span\{[0\ 0\ 1\ 1]^T\}$.

(c) Can we find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? . . . . (6p)
Answer: Yes, for example $u_1 = -x_3,\ u_2 = -2x_4$.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -3 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ \alpha & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x, \end{aligned}$$

where α is a real constant.

(a) For what values of α is the noninteracting control problem solvable? . . . . . (4p)
Answer: $\alpha \neq \frac{1}{2}$.

(b) What is the zero dynamics of the system when the noninteracting control problem is solvable? . . . . (6p)
Answer: $\dot{z} = -3z$.

(c) Suppose now that the second output y_2 is taken away from the system, namely only y_1 is kept. What is the (transmission) zero(s) of the system now? . (10p)
Answer: 1. $\alpha \neq \frac{1}{2}$. From (b) we already know that $P_\Sigma(s)$ has rank 5 if $s \neq -3$, and we can verify that $P_\Sigma(-3)$ has also rank 5. Thus there is no zero. 2. $\alpha = \frac{1}{2}$. We can consider $\frac{1}{2}u_1 + u_2$ as one control, then we have zeros as the roots of $2s^2 + 5s + 1 = 0$.

4. Consider:

$$\begin{aligned} \dot{x} &= Ax + bu + Pw \\ \dot{w} &= \Gamma w \\ y &= cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 9 & 0 \\ -1 & -\alpha & -1 \\ 0 & 0 & -\alpha \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \ P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \ c = \begin{pmatrix} 3 & 0 & 0 \end{pmatrix}$$

and α is a real constant.

(a) When $\alpha > 0$, show the **existence** of an invariant subspace $x = \Pi(\alpha)w$ for the system when u is set to zero (no exact Π is needed) . . . . (5p)
Answer: In this case the A matrix is a stable matrix, thus ...

(b) Find the values of α, such that the reduced system

$$\begin{aligned} \dot{w} &= \Gamma w \\ y &= c\Pi(\alpha)w \end{aligned}$$

is not observable, where Π is given in (a). . . . . (8p)
Answer: (c, A) and $([1\ 0], \Gamma)$ are observable. The reduced system loses observability if a transmission zero (where the "b" vector is $[1\ 0\ 1]^T$) is either 0 or 2, which implies $\alpha = -5, \pm 3, 1$.

(c) Show for almost all values of α, the full information output regulation problem with the tracking error $e = y - w_2$ is solvable (*a solution is not required*) . What are the values of α that may make the problem unsolvable? . . . . (7p)
Answer: Controllability requires $\alpha \neq -9$. The transmission zero requirement (now we use the b vector) implies that $\alpha \neq 1, 3$.

5. Consider in a neighborhood N of the origin

$$\begin{aligned} \dot{x}_1 &= x_2 + \sin(x_3) \\ \dot{x}_2 &= x_1^3 + \cos(x_3)u \\ \dot{x}_3 &= -\alpha x_1^3 - \sin(x_3) - \sin(x_3)u \\ y &= x_2, \end{aligned}$$

where α is a real constant.

(a) Convert the system into the normal form. . . . . (6p)

Answer: The system has relative degree 1 at the origin. Let $z_1 = x_1, z_2 = e^{x_2}\sin(x_3), \xi = x_2$, we have $\dot{z}_1 = e^{-\xi}z_2 + \xi, \dot{z}_2 = z_1^3 z_2 - \alpha z_1^3 e^{\xi}\cos(x_3) - \cos(x_3)z_2, \dot{\xi} = -z_1^3 + \cos(x_3)u, \cos(x_3) = \sqrt{1 - e^{-2\xi}z_2^2}$.

(b) Analyze the stability of the zero dynamics in terms of α. . . . . (4p)

Answer: The zero dynamics is asymptotically stable if $\alpha > 0$, unstable if $\alpha < 0$, critically stable if $\alpha = 0$.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. . . . . (4p)

Answer: Any controller that makes $\dot{\xi} = -k\xi$, where $k > 0$.

(d) Is the system without the output exactly linearizable? . . . . (6p)

Answer: No, since the linearized system is not controllable.