KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

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Allowed written material: the lecture notes, the exercise notes, your own class notes and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 45 points give grade E, 50 points D, 61 points C, 76 points B, and 91 points A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$.

If $V^* = 0$ for system (1), then $(C, A + BF)$ is observable for any F. ..... (5p)

Answer: Ture, since $ker\ \Omega_F \subset V^* = 0$ for any F.

(b) Let $p = m$. If system (1) has degree $(r_1, \cdots, r_m)$ and $\sum_{i=1}^m r_i = n$, then it does not have any (transmission) zero. .... (5p)

Answer: Ture, since $V^* = 0$ in this case and the number of zeros equal the dimension of V^* minus the dimension of R^*.

(c) Consider a controllable and observable (when w is set to 0) linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx, \end{aligned}$$

where w is disturbance. If (A, E) is also controllable, then $Im\ E$ is not contained in V^*. .... (5p)

Answer: False, since $(A+BF, E)$ may not be controllable for some F although $(A + EK, E)$ is always controllable.

(d) Consider a nonlinear single-input system

$$\dot{x} = f(x) + g(x)u$$

where $x \in R^n$, f, g can be differentiated infinitely many times and $f(0) = 0$. Let $A = \frac{\partial f}{\partial x}|_{x=0}$ and $b = g(0)$.

A necessary condition for the existence of an output mapping $h(x)$ such that the system has relative degree n at the origin is that (A, b) is controllable. (5p)

Answer: True, since the existence of such $h(x)$ would imply that the system is exactly linearizable.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} u \\ y &= (0\ 0\ 0\ 1)x. \end{aligned}$$

(a) Find V^*. .... (8p)

Answer: $V^* = \{x : x_3 = x_4 = 0\}$: $\Omega_0 = sp\{(0\ 0\ 0\ 1)\}$, $\Omega_0 \cap G^\perp = sp\{(0\ 0\ 0\ 1)\}$, and $(0\ 0\ 0\ 1)A = (0\ 0\ 1\ \ -1)$, then $\Omega_1 = sp\{(0\ 0\ 0\ 1); (0\ 0\ 1\ \ -1)\}$. Since $\Omega_2 = \Omega_1$, we have $V^* = \Omega_1^\perp$.

(b) Find R^*. .... (6p)

Answer: $R^* = \{x : x_1 = x_3 = x_4 = 0\}$: A simple friend is $u_1 = f_1 x = 0$, $u_2 = f_2 x = -x_2$, then $R^* =< A + BF | V^* \cap ImB >= V^* \cap ImB$.

(c) Can we find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? .... (6p)

Answer: Yes, we can easily find an F to modify all the eigenvalues to -1 for example (what will happen if a_{11} is 1 instead?)

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 1 & 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ \alpha & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x, \end{aligned}$$

where α is a constant.

(a) What are the zeros? .... (6p)

Answer: $s_0 = -2$. When $\alpha \neq 0$, the system has relative degree $(1, 2)$ and the zero dynamics is $\dot{z} = -2z$. When $\alpha = 0$, only u_2 is active, we can use the previous case the establish that only $s_0 = -2$ may make the system matrix lose rank, and this is verified.

(b) For what α is the noninteracting problem solvable? ....(4p)
Answer: $\alpha \neq 0$, which implies that the system has relative degree.

(c) Now suppose one of the two output sensors is out of order, namely we have only one output for the system y_2 available. What are the zeros now? .. (10p)
Answer: $s_0 = 1, -2$ if $\alpha = 0$, otherwise there is no zero. When $\alpha = 0$ the system is reduced to a SISO system that has relative degree 2. The zero dynamics is $\dot{z}_1 = -2z_1 + z_2, \dot{z}_2 = z_2$. When $\alpha \neq 0$,only $s_0 = -2$ may make the system matrix lose rank (from discussion in (a)), and we can verify that the matrix has rank 5 at -2.

4. Consider:

$$\begin{aligned} \dot{x} &= Ax + bu + pqw \\ \dot{w} &= \Gamma w \\ y &= cx, \end{aligned}$$

where α is a positive constant, w represents both disturbances and reference output signals.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -4 & 0 \\ 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 \end{pmatrix} \; b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \; p = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 0 \end{pmatrix} \; q = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \; c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Some facts you can use: $eig(A) = \{-0.82 \pm 1.8i, -1.68 \pm 0.87i\}$, (A, p) is controllable.

(a) Show the **existence** of an invariant subspace $x = \Pi(\alpha)w$ for the system when u is set to zero (no exact Π is needed) . ... (5p)
Answer: Since A is stable and Γ is antistable, there exists such an invariant subspace.

(b) Show for almost all values of α, the reduced system

$$\begin{aligned} \dot{w} &= \Gamma w \\ y &= c\Pi(\alpha)w \end{aligned}$$

is observable, where both Γ and c are defined as above. What is the value of α that makes the reduced system unobservable?............................(8p)
Answer: The reduced system is observable if no eigenvalue of Γ is a transmission zero of $\dot{x} = Ax + pu, \; y = cx$, since (A, p, c) is minimal. The transmission zeros are 2 and -1. Therefore, for $\alpha = 2$, the reduced system is not observable.

(c) For $\alpha = 2$, solve the full information output regulation problem (find u), where the tracking error is

$$e = y - (0 \; 1 \; 0)w.$$

Answer: Let $u = K(x - \Pi w) + dw$ (the letter Γ is already taken) , we can let $K = 0$ since A is already stable. In order to find d we need to find Π anyway. Let $\Pi = (\pi_1; \pi_2; \pi_3; \pi_4)$. From $c\Pi - q = 0$ we have $\pi_1 = q$, from $\pi_2 w = \dot{(\pi_1 w)}$, we have $\pi_2 = \pi_1 \Gamma$. Similarly, from $\pi_2 \Gamma = -\pi_1 - 2\pi_2 - 4\pi_3 + (1\ 0\ 0)$ we can solve for π_3. Finally we can solve π_4 from $\pi_4 \Gamma = \pi_3 - 2\pi_4$. This Sylvester equation has a unique solution since Γ does not have any eigenvalue with negative real part. When Π is solved we can solve d from $\pi_3 \Gamma = \pi_1 - 3\pi_3 - 4\pi_4 + d + (1\ 0\ 0)$. One can also first convert the system without w into the normal form with relative degree 3. ....(7p)

5. Consider in a neighborhood N of the origin

$$\begin{aligned}
\dot{x}_1 &= x_2 + \sin(x_3) \\
\dot{x}_2 &= x_1^4 + x_2 + u \\
\dot{x}_3 &= \alpha x_1^3 - \sin(x_3) + \cos(x_2) u \\
y &= \sin(x_2),
\end{aligned}$$

where α is a real constant.

(a) Convert the system into the normal form. ....(6p)

Answer: Let $\xi = \sin(x_2), z_1 = x_1, z_2 = x_3 - \sin(x_2)$, then in the normal form we have

$$\begin{aligned}
\dot{z}_1 &= x_2 + \sin(x_3) \\
\dot{z}_2 &= \alpha x_1^3 - \sin(x_3) - \cos(x_2)(x_1^4 + x_2) \\
\dot{\xi} &= \cos(x_2)(x_1^4 + x_2 + u) \\
y &= \xi,
\end{aligned}$$

(one is encouraged to reexpress the right hand side with new coordinates)

(b) Analyze the stability of the zero dynamics in terms of α. ....(4p)

Answer: Since the zero dynamics is

$$\begin{aligned}
\dot{z}_1 &= \sin(z_2) \\
\dot{z}_2 &= \alpha z_1^3 - \sin(z_2) - z_1^4
\end{aligned}$$

By using center manifold theory, where $h(z_1) \approx \alpha z_1^3$ if $\alpha \neq 0$ and $h(z_1) \approx -z_1^4$ if $\alpha = 0$, we establish that it is asymptoitcally stable if $\alpha < 0$; unstable if $\alpha \geq 0$.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. ....(4p)

Answer: We can design u by for example let $\dot{\xi} = \cos(x_2)(x_1^4 + x_2 + u) = -\xi$.

(d) Is the system without the output exactly linearizable? ....(6p)

Answer: No since the linearization is not controllable.