KTH Matematik

SF2842: Geometric Control Theory

Homework 1

Due February 11, 16:50pm, 2016

You may use min(5,(your score)/4) as bonus credit on the exam

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1. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -2 & 0 & 0 & -1 \\ 0 & -2 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} u \\ y &= (1\ 1\ 0\ 0)x.\end{aligned}$$

(a) Compute $\mathcal{V}^*$ and express all friends F of $\mathcal{V}^*$. . . . . (2p)

Solution: $V^* = \text{span}\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\}, \mathcal{F}(V^*) = \{F \in R^{2\times 4} | f_{22} - f_{21} = 1, f_{24} - f_{23} = \frac{1}{2}\}$

(b) Compute $\mathcal{R}^*$ that is contained in $ker\ C$. . . . . (2p)

Solution: $R^* = \text{span}\{\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\} = V^*$

(c) Can we find a friend F of $\mathcal{V}^*$ such that $(A + BF)$ has all eigenvalues with negative real parts? . . . . (3p)
Solution: Yes, since (A, B) is reachable and $E = 0 \in \text{Im } R^*$. According to the theorem 4.3 in the compendium, the pole assignment problem can always be solved.

2. Consider

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx,\end{aligned}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$.

(a) Show the controllable subspace is (A+BF)-invariant for any F. . . . . (2p)
Solution: For any vector $v \in R = \langle A|\mathrm{Im}B\rangle$ there exist $\alpha_1, \cdots, \alpha_n$ and $v_1, \cdots, v_n$ such that

$$v = \sum_{i=1}^{n} \alpha_i A^{i-1} B v_i.$$

For any F, we have

$$(A + BF)v = \sum_{i=1}^{n} \alpha_i (A^i B + BFA^{i-1}B)v_i \in R,$$

since $A^i B v_i \in R$ and $BFA^{i-1}Bv_i \in \mathrm{Im}B \subset R$, for $i = 1, 2, \cdots, n$. Hence, R is $(A + BF)$-invariant for any F.

(b) Assume further that $CA^kB \neq 0$, for some $k < n$, and (C, A) is not observable. Show the unobservable subspace $ker\ \Omega$ is not (A+BF)-invariant for all F.(3p)
Solution: Suppose $\ker\Omega$ is $(A + BF)$-invariant for any F, and v is a nonzero vector in $\ker\Omega$. We know that $(A + BF)v \in \ker\Omega$, which is equivalent to $\Omega(A + BF)v = 0$. By the definition of Ω, we get $CA^i(A + BF)v = 0$ for $i = 0, 1, \cdots, n-1$. $v \in \ker\Omega$ means that $CA^iv = 0$ for $i = 0, 1. \cdots, n$, which gives $CA^iBFv = 0$ for $i = 0, 1. \cdots, n-1$. Especially, we have $CA^kBFv = 0$. Since $CA^kB \neq 0$ and $v \neq 0$, we can always find a matrix F such that $CA^kBFv \neq 0$, which makes a contradiction.

(c) Suppose (C, A) is observable and the dimension of $\mathcal{V}^*$ is greater or equal to one. Show it is not possible to express a friend F of $\mathcal{V}^*$ as $F = LC$, namely it is not possible to use output feedback to make $\mathcal{V}^*$ invariant. . . . . (2p)
Solution: Suppose $F = LC$ is a friend of V^* and v is a nonzero vector in V^*. Then we have $(A + BLC)v \in V^* \subset \ker C$. Since $v \in V^* \subset \ker C$, $Cv = 0$. So we get $Av \in V^*$. We can continue the similar derivation to get $A^iv \in V^* \subset \ker C$, for $i = 0, 1, \cdots, n-1$. This implies v is a nonzero vector in $\ker\Omega$, which contradicts the assumption that (C, A) is observable.

3. Consider

$$\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 + x_3 + x_4 \\
\dot{x}_2 &= -x_1 - \alpha u \\
\dot{x}_3 &= -x_2 - 2x_3 + u \\
\dot{x}_4 &= x_2 - u \\
y &= x_3 + x_4,
\end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form and compute the zero dynamics. (2p)
Solution: Normal form:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 & \frac{\alpha}{2} \\ -1 & -\alpha & 0 & \alpha - \frac{\alpha^2}{2} \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 2(\alpha - 1) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} u$$

$$y = \xi_1,$$

where $z_1 = x_1$, $z_2 = x_2 + \alpha x_3$, $\xi_1 = x_3 + x_4$, and $\xi_2 = -2x_3$.
Zero dynamics:

$$\dot{z} = Nz, \text{ where } N = \begin{pmatrix} -1 & 1 \\ -1 & -\alpha \end{pmatrix}.$$

(b) Computer $\mathcal{V}^*$ and $\mathcal{R}^*$ in $ker\ C$. . . . . (2p)

Solution: $V^* = \text{spam}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\}$, $R^* = \{0\}$.

(c) For what α we can find a friend f of $\mathcal{V}^*$ such that $(A + bf)$ is a stable matrix? (2p)
Solution: It is only when the zero dynamics is stable can we stabilize the system with a friend of V^*, which is when $\alpha > -1$.