Exam May 27 2014 in SF2852 Optimal Control.

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Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Let

$$\begin{aligned} T &= \text{Temperature of an object} \\ P &= \text{Power of a heating source} \end{aligned}$$

We assume that the temperature satisfies

$$\dot{T} = P - T, \qquad T(0) = 0$$

Find the optimal power supply $0 \leq P(t) \leq 2$ such that $T(1) = 1$ and

$$\int_0^1 P(t)dt$$

is minimized. .... (10p)

2. Determine the optimal control for the following two problems. Note that in the second problem time is also a free variable.

(*a*)

$$\min_{u(\cdot)} \frac{1}{2}\int_0^T (1 + (1+t)u(t)^2)dt, \text{ subj. to } \begin{cases} \dot{x}(t) = u(t), \\ x(0) = x_0 \\ x(T) = 0 \end{cases}$$

.... (5p)

(*b*)

$$\min_{u(\cdot), T\geq 0} \frac{1}{2}\int_0^T (1 + (1+t)u(t)^2)dt, \text{ subj. to } \begin{cases} \dot{x}(t) = u(t), \\ x(0) = x_0 \\ x(T) = 0, \ T \geq 0 \end{cases}$$

.... (5p)

3. Consider the following shortest path problem

$$\min \Sigma_{k=0}^{5} f_0(x_k, u_k) \text{ subj. to } \begin{cases} x_{k+1} = x_k + u_k \\ x_0 = (0,0) \\ x_6 = (3,3) \\ u_k \in \{(1,0),(0,1)\} \end{cases}$$

which corresponds to computing the shortest path from the node $(0,0)$ to node $(3,3)$ in the graph in Figure 1.

- The state space of the system are the nodes (k,l) $k,l = 0,1,2,3$ in Figure 1.
- The costs $f_0(x_k, u_k)$ are indicated on the edges in Figure 1.
- The control $u = (1,0)$ means "go down" in the graph, and the control $u = (0,1)$ means "go right" in the graph. If you reach either of the boundaries (the nodes $(k,3)$ and $(3,l)$) then you have only one feasible control choice left.
- Note that the terminal state constraint simplifies the calculations further.

(*a*) Formulate the problem as a discrete time optimal control problem on the following form

$$\min \phi(x_N) + \Sigma_{k=0}^{N-1} f_0(x_k, u_k) \text{ subj. to } \begin{cases} x_{k+1} = f(x_k, u_k) \\ x_0 \text{ given} \\ u_k \in U(x_k) \end{cases}$$

.... (5p)

(*b*) Solve the problem using dynamic programming. .... (5p)

Remark: Note that here as well as in all dynamic programing approaches you create solutions to the whole family of similar problems having different initial states. Note also that the "backward" direction of solving the problem by dynamic programming gives a time complexity of n^2 compared to 2^{2n} for an exhaustive search (n being the side of the square).

4. Consider the linear quadratic control problem

$$V(x_0) = \min_{u_k, k=0,\dots,N-1} x_N^T Q_0 x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k)$$
$$\text{subject to} \quad x_{k+1} = Ax_k + Bu_k,\ x(0) = x_0,$$

where Q_0, Q, and R are symmetric positive definite matrices.

(a) Show that the optimal cost is $V(x_0) = x_0^T P_0 x_0$, where P_0 is determined by the discrete-time Ricatti equation:

$$P_N = Q_0,$$
$$P_k = Q + A^T(P_{k+1} - P_{k+1}B(R + B^T P_{k+1}B)^{-1}B^T P_{k+1})A,$$

for $k = N-1, N-2, \dots, 0$. .... (7p)

(b) Determine the optimal feedback $u(x)$. .... (3p)

5. Consider the minimum time problem

$$\min T \text{ subject to } \begin{cases} \dot{x} = -x + u, & x(0) = x_0, \quad x(T) = 0 \\ \dot{y} = u, & y(0) = y_0, \quad y(T) = 0 \\ |u(t)| \le 1. & \end{cases} \tag{1}$$

- (a) Show that the optimal control in bang-bang with at most one switch. .... (3p)
- (b) Determine the set from which the $[0,0]^T$ could be reached without any switch (i.e., the switching curve). .... (3p)
- (c) Plot the phase diagram and determine the optimal feedback law. .... (4p)

Solutions

1. The optimal control problem becomes

$$\min \int_0^1 P(t)dt \quad \text{subj. to} \quad \begin{cases} \dot{T} = P - T, & T(0) = 0,\ T(1) = 1 \\ 0 \le P \le 2 \end{cases}$$

The Hamiltonian becomes

$$H(T, P, \lambda) = P + \lambda(P - T)$$

Pointwise minimization gives

$$P^* = \text{argmin}_{0 \le P \le 2} P + \lambda(P - T) = \begin{cases} 0, & \sigma(t) > 0 \\ 2, & \sigma(t) < 0 \end{cases}$$

where $\sigma(t) = 1 + \lambda(t)$. The adjoint equation

$$\dot{\lambda} = \lambda,$$

has the solution $\lambda(t) = e^t\lambda_0$ and thus $\sigma(t) = 1 + e^t\lambda_0$. If $\lambda_0 > 0$ then $\sigma(t) > 0$ for all t which is impossible. For $\lambda_0 < 0$ we get

$$P^* = \begin{cases} 0, & 0 \le t \le t_1 \\ 2, & t_1 \le t \le 1 \end{cases}$$

where t_1 is determined by the condition $T(1) = 1$, i.e.,

$$T(1) = \int_{t_1}^{1} e^{-(1-s)} 2ds = (1 - e^{-(1-t_1)})2 = 1$$

which gives $t_1 = 1 - \ln 2$.

2. (*a*) Since, T is fixed the problem simplifies to

$$\min_{u(\cdot)} \frac{1}{2}\int_0^T (1+t)u^2 dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = u, \ x(0) = x_0 \\ x(t_f) = 0 \end{cases}$$

Let $H(t, x, u, \lambda) = \frac{1}{2}(1+t)u^2 + \lambda u$. Pointwise minimization gives

$$\mu(t, x, \lambda) = \operatorname{argmin} H(t, x, \lambda) = -\frac{\lambda}{1+t}$$

This gives the TPBVP

$$\begin{aligned} \dot{x} &= -\lambda, \qquad x(0) = x_0, \ x(t_f) = 0 \\ \dot{\lambda} &= 0, \qquad \lambda(t_f) = free \end{aligned} \tag{2}$$

This gives $u_{opt} = -\lambda_0/(1+t)$ where λ_0 is constant and can be determined from

$$\begin{aligned} &0 = x(t_f) = x_0 - \int_0^T \frac{\lambda_0}{1+t} = x_0 - \lambda_0 \log(1+T) \\ &\Rightarrow \lambda_0 = x_0/\log(1+T). \end{aligned}$$

Hence, $u_{opt}(t) = -x_0/(1+t)/\log(1+T)$.

(*b*) The Hamiltonian becomes $H(t, x, u, \lambda) = \frac{1}{2}(1 + (1+t)u^2) + \lambda u$. Pointwise minimization and the Hamiltonian system is the same as in (2). To find the optimal transition time we either optimize the cost or use the condition on $H^*(t) = H(t, x^*, \mu(t, x^*, \lambda), \lambda)$ in

PMP. Here we use that $H(T, x^*(T), \mu(T, x^*(T), \lambda(T)), \lambda(T)) = 0$. This gives

$$
\begin{aligned}
0 &= H(T, x^*(T), \mu(T, x^*(T), \lambda(T)), \lambda(T)) \\
&= \frac{1}{2}(1 + (1+T)u_{\text{opt}}^2(T)) + \lambda(T)u_{\text{opt}}(T) \\
&= \frac{1}{2}(1 - x_0^2/(1+T)/\log(1+T)^2),
\end{aligned}
$$

hence the optimal T is given by $x_0 = (1+T^*)\log(1+T^*)^2$. The optimal control is then

$$u_{opt} = u_{opt}(t) = -x_0/(1+t)/\log(1+T^*).$$

3. This problem is solved in a similar way as the shortest path problem in the lecture notes. Note that the constraint $x_6 = (3,3)$ has to be represented using either the feasible control set $(U(x_k))$ or using the terminal cost $(\phi(x_6))$. The optimal cost (shortest path) is 16.

4. See Example 7 on page 23-24 in the course book.

5. (a) The hamiltonian is given by $H(x, u, \lambda) = 1 + \lambda^T(Ax + bu)$ where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Pointwise minimization gives

$$\mu(t, x, \lambda) = \operatorname{argmin} H(t, x, \lambda) = -\operatorname{sign}(\lambda_1 + \lambda_2), \text{ for } \lambda_1 + \lambda_2 \neq 0.$$

The adjoin system is

$$\lambda = -\frac{\partial H}{\partial x} = -A^T\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \tag{3}$$

hence $\lambda_1 = e^t c_1$ and $\lambda_2 = c_2$ where c_1, c_2 are constants. Since the Hamiltonian is zero along the optimal path, both $c_1 = c_2 = 0$ is impossible. Therefore $\lambda_1 + \lambda_2 = e^t c_1 + c_2$ is either strictly increasing, strictly decreasing, or non-zero constant, and hence the control is bang-bang with at most one switch.

(b) Consider the system trajectory when $u = 1$. Then

$$
\begin{aligned}
0 &= y(t) = t + y(0) \\
0 &= x(t) = e^{-t}x(0) + \int_0^t e^{t-\tau}d\tau = e^{-t}(x(0) - 1) + 1
\end{aligned}
$$

whenever $y(0) = -t$ and $x(0) = 1 - e^t$. Similarly, the control $u = -1$ takes the system to 0 when

$$\begin{aligned} 0 &= y(t) = -t + y(0) \\ 0 &= x(t) = e^{-t}x(0) - \int_0^t e^{t-\tau}\,d\tau = e^{-t}(x(0)+1)+1 \end{aligned}$$

when $y(0) = t$ and $x(0) = e^t - 1$. The switching curve is thus given by $(x, y) = (\text{sgn}(t)(e^{|t|} - 1), t)$.

(c) Draw the phase diagram and note that the control $u = 1$ must be used when the state on the right side of the switching curve and the control $u = -1$ is used when the state is on the left side of the switching curve. That is,

$$u^*(x,y) = \begin{cases} -1 & \text{when} \quad x(t) < \text{sign}(y)(e^{|y|} - 1) \\ -\text{sign}(y) & \text{when} \quad x(t) = \text{sign}(y)(e^{|y|} - 1) \\ 1 & \text{when} \quad x(t) > \text{sign}(y)(e^{|y|} - 1). \end{cases}$$

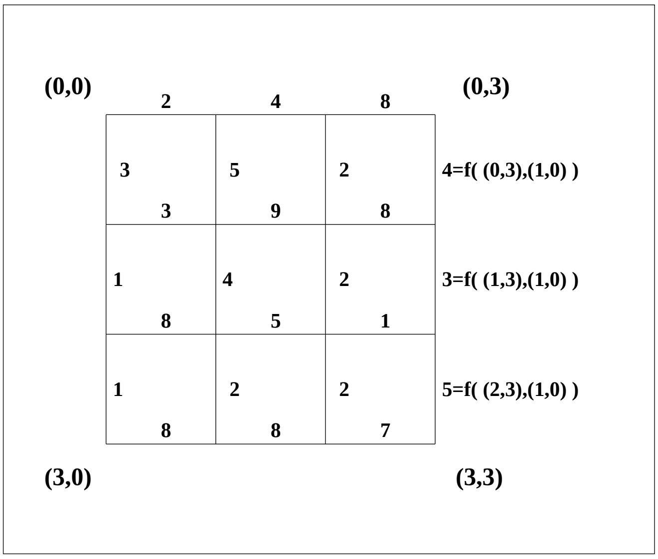

Figure 1: The discrete optimal control problem. The small font numbers being $f_0(x_k, u_k)$ and the large ones being states (e.g. (0,0))